1 Basic set theory

1.1 What is a set, and how do you name one?

- A set is an abstract collection of distinct objects, called its *members*.

  A set can contain any kind of object at all. For example, we can have a set that contains the number 5, a camel, and the set of all classics professors. Since a set is a collection of objects, one set cannot contain a single object ‘twice’, any more than a bushel of apples can contain a single apple twice.

- Sets are defined and distinguished only by what members they contain, nothing else.

  The way we describe the set is irrelevant to its identity. One set may have many names. The following descriptions, for example, name the same set: “The set containing the number 5” “The set containing the third prime number”, “The set containing the integer immediately preceding 6”.

- The elements of sets are not ordered with respect to one another.

  When you name a set, the order in which you name the objects it contains is therefore irrelevant.

- The usual way to name a set is by giving a description of its members inside curly brackets, e.g.:

  \[
  \{ \text{George Bush, Dick Cheney} \} 
  \]
Often we describe the members of a set by listing them, as above. Or we can denote them by giving a property which they (and only they) have. This is often done as follows:

\[
\{ x \mid \ldots \} 
\]

Read the bar as "such that". Read the entire expression as: “The set of all \( x \) such that \( \ldots \).” The dots are to be filled in by a description of the property which all the \( x \)'s must have. Example:

\[
\{ x \mid x \text{ is either the 43rd US president or his vice president} \}
\]

- The unique set which contains no members is called “the empty set”, written “\( \emptyset \)”.

1.2 Sequences (not sets)

- **Sequence**: A list of \( n \) ordered objects.

Sequences (also called *tuples*) are written either inside regular parentheses, \((x,y,\ldots)\), or inside angle brackets, \(<x,y,\ldots>\).

- Notice: \((2,3) \neq (3,2), (a,j,5) \neq (5,j,a), \text{etc.}\)

- Sequences may include the same element more than once, e.g.: \((2,2)\).

- A tuple with two elements is called an ordered pair.

1.3 Set-theoretic relations

1. **Membership**: \( \in \)

- Membership is a one-way relation between an object (of whatever kind) and a set.
- “\( x \in A \)” is read as: “\( x \) is a member of \( A \)”.
- **Definition**: Object \( x \) is a member of set \( A \) just when \( A \) immediately contains the object \( x \). Please note, the membership relation is not transitive!
• *Examples*
  5 ∈ \{ George Bush, 5 \}
  Dick Cheney \∉ \{ George Bush, 5 \}
  \{ 5 \} ∈ \{ George Bush, \{ 5 \}\}
  \{ 5 \} \not\in \{ George Bush, 5 \}
  5 \not\in \{ George Bush, \{ 5 \}\}

2. **Subset:** \(\subseteq\)

• The subset relation is one-way relation between two sets.
• “\(A \subseteq B\)” is read as: “\(A\) is a subset of \(B\)”.
• *Definition:* \(A\) is a subset of \(B\) just when every member of \(A\) is also a member of \(B\).
• *Examples*
  \{ 5 \} \subseteq \{ George Bush, 5 \}
  \{ 5, George Bush \} \subseteq \{ George Bush, 5 \}
  5 \not\in \{ George Bush, 5 \}
  \{ Dick Cheney \} \not\in \{ George Bush, 5 \}
  \{ George Bush, 5, Dick Cheney \} \not\in \{ George Bush, 5 \}
  \emptyset \subseteq \{ George Bush, 5 \}

3. **Proper subset:** \(\subset\)

• The proper-subset relation is one-way relation between two sets.
• “\(A \subset B\)” is read as: “\(A\) is a proper subset of \(B\)”.
• *Definition:* \(A\) is a subset of \(B\) just when every member of \(A\) is also a member of \(B\), and also \(A\) and \(B\) are not equal.
• *Examples*
  \{ 5 \} \subset \{ George Bush, 5 \}
  \{ George Bush \} \subset \{ George Bush, 5 \}
  \emptyset \subseteq \{ George Bush, 5 \}
  \{ George Bush, 5 \} \not\in \{ George Bush, 5 \}
1.4 Set-theoretic operations

1. **Union**: $\cup$
   - The union operation takes two sets as input, and outputs a set.
   - "$A \cup B$" is read: "The union of $A$ and $B$"
   - **Definition**: The set $A \cup B$ is the set of all objects which are *either* members of $A$ or members of $B$ (or of both).
   - **Examples**:
     \[
     \begin{align*}
     \{a, j, 5\} \cup \{m\} &= \{a, j, 5, m\} \\
     \{a, j, 5\} \cup \{a, 5\} &= \{a, j, 5\} \\
     \{a, j, 5\} \cup \emptyset &= \{a, j, 5\}
     \end{align*}
     \]

2. **Intersection**: $\cap$
   - The intersection operation takes two sets as input, and outputs a set.
   - "$A \cap B$" is read: "The intersection of $A$ and $B$.”
   - **Definition**: The set $A \cap B$ is the set of all objects which are *both* members of $A$ and members of $B$.
   - **Examples**:
     \[
     \begin{align*}
     \{a, j, 5\} \cap \{m\} &= \emptyset \\
     \{a, j, 5\} \cap \{a, 5\} &= \{a, 5\} \\
     \{a, j, 5\} \cap \emptyset &= \emptyset
     \end{align*}
     \]

3. **Set complementation**: $-$
   - The set complementation operation takes an ordered pair of sets as input, and outputs a set.
   - "$A - B$" is read: "A minus B" or "The complement of B in A”.
   - **Definition**: The set $A - B$ is the set of all object that are members of $A$ but not members of $B$.
   - **Examples**:
     \[
     \begin{align*}
     \{a, j, 5\} - \{a, j\} &= \{5\} \\
     \{a, j\} - \{a, j, 5\} &= \emptyset \\
     \{a, j\} - \emptyset &= \{a, j\}
     \end{align*}
     \]

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4. **General complementation :** ' 

- The general complementation operation takes a single set as input, and outputs a set.
- “A′” is read: “The complement of A.”
- **Definition:** The set \(A'\) is the set of all objects in the domain which are not members \(A\). The domain is the set of all objects presumed to exist.
- **Examples**
  - Let the domain be \(D\), and: \(D = \{a, j, 5\}\). And let \(A = \{a\}\). Then: \(A' = \{j, 5\}\).
  - Let the domain be \(D\), and: \(D = \{a, j, 5\}\). And let \(A = \{a, j, 5\}\). Then: \(A' = \emptyset\).

*Notice:* All of this can be conceived very easily in terms of Venn diagrams, such as those on John Bonham’s bass drum.

5. **Power set :** \(\varphi\) 

- The powerset operation takes a single set as input, and outputs a set.
- “\(\varphi(A)\)” is read: “The powerset of \(A\)”.
- **Definition:** The set \(\varphi(A)\) is the set of all sets which are subsets of \(A\).
- **Example:** 
  \(\varphi(\{a, j, 5\}) = \{\emptyset, \{a\}, \{j\}, \{5\}, \{a, j\}, \{j, 5\}, \{a, 5\}, \{a, j, 5\}\}\)

6. **Cartesian product :** \(\times\) 

- The Cartesian product operation takes an ordered pair of sets as input, and outputs a set.
- “\(A \times B\)” is read: The (Cartesian) product of \(A\) and \(B\).
- **Definition:** The set \(A \times B\) is the set of all ordered pairs \((x, y)\) such that \(x\) is a member of \(A\) and \(y\) is a member of \(B\).
- **Examples:** 
  \(\{a,j\} \times \{5\} = \{(a,5), (j,5)\}\)
  \(\{a,j\} \times \{m,5\} = \{(a,m), (a,5), (j,m), (j,5)\}\)
• **N.B.** The product operation can be repeated \( n \) times, giving ‘\( n \)-tuples’ rather than just pairs. Terminologically, \( A \times A \times A \ldots, n \) times, is written: \( A^n \).

7. **Cardinality**

• The cardinality of a set is the number of members it has.
• “\(|A|\)” is read: “the cardinality of \( A \)”.
• Examples:
  \(|\{a, j, 5\}| = 3\)
  \(|\emptyset| = 0\)

1.5 **Questions**

*Answer true or false to the following statements.*

1. \( \{a\} \in \{a, b, c\} \)
2. \( \{a, b, c\} = \{b, a, c\} \)
3. \( \{a\} \subset \{a\} \)
4. \( \{a, \{a\}\} \subset \{a, b, c\} \)
5. \( 5 \in \{5, 6, 7\} \)
6. \( \{Superman\} = \{Clark Kent\} \)
7. \( \emptyset \in \{5, Joe Camel\} \)
8. \( \emptyset \subset \{5, Joe Camel\} \)
9. \( (A \times A) \subseteq A \)
10. \(|A \times A| > |A|\)

2 **Relations**

2.1 **Defining relations**

• We speak of relations between objects, such as the relation of *being taller than* or the relation *loving*.

• Call a relation which relates \( n \) objects an “\( n \)-place relation”. Call a 1-place relation a “property.”
• Formally we will define an relation as follows:

**Relation** An n-place relation among members of a set A is a subset of $A^n$.

**Property** A 1-place relation, or property, in the set A is a subset of A.

• **Examples**
  Suppose: the set of boys, $B = \{ \text{Al, Bill, Cal} \}$;  
  the set of girls, $G = \{ \text{Dana, Emma} \}$;  
  Al and Cal love Emma, Dana loves Bill, and that’s it.

  Then: the property of *being a boy* is defined as $B$;  
  the property of *being a girl* is defined as $G$;  
  the relation of loving, $L$, is defined as:  
  $L = \{ (\text{Al, Emma}), (\text{Cal, Emma}), (\text{Dana, Bill}) \}$.  
  (Notice, $L \subseteq (B \cup G)^2$.)

• When relation $R$ has the n-tuple $(\ldots)$ as a member, we write $R(\ldots)$, and say “$R$ holds of $(\ldots)$”, or “$R$ of $(\ldots)$”.

2.2 Properties of relations

1. **Reflexivity**: $R$, where $R \subseteq A^2$ is reflexive when every member of $A$ has the $R$-relation to itself.
   *Example* Among numbers, the relation “$\geq$” is reflexive.

2. **Symmetry**: $R$, where $R \subseteq A^2$, is symmetric when $R(x, y)$ if and only if $R(y, x)$.
   *Example* The relation of “being a sibling of” is symmetric.

3. **Transitivity**: $R$, where $R \subseteq A^2$, is transitive when $R(x, y)$ and $R(y, z)$ always means that $R(x, z)$.
   *Example* The relation of “being an ancestor of” is transitive.

4. **Functionhood**: $R$, where $R \subseteq A^2$, is a function when no member of $A$ is $R$-related to more than one object.
   *Example* The relation between objects and their ages is a function.
2.3 Graphs

- A graph is a pair \((V, E)\).
- \(V\) is a set of objects, which we regard as vertices.
- \(E\) is a subset of \(V \times V\), a relation among vertices. which we regard as edges connecting the vertices.
- In a directed graph, \(E\) is not presumed to be a symmetric relation.
- In an undirected graph, \(E\) is presumed to be symmetric.

3 Basic propositional logic

- We will define propositions as those objects which can be interpreted as true or false.\(^1\)
- Various operations (‘connectives’) can be defined which take one or more propositions as input, and output a proposition. Below are some Boolean operations.

\[ \land = \text{‘and’} \quad \lor = \text{‘or’} \quad \neg = \text{‘not’} \]

*Example:* \((P \land \neg Q) \lor T = \text{‘Either } P \text{ and } \neg Q \text{ or } T'.\)

- The meaning of these operations is defined, completely, by listing what output they yield for each possible input.

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<th>(P)</th>
<th>(Q)</th>
<th>(P \land Q)</th>
<th>(P)</th>
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<th>(P \lor Q)</th>
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<th>(\neg P)</th>
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- Another operation on propositions is logical implication: \(\rightarrow\), defined as follows:

\(^1\)Notice, we sometimes use “proposition” to mean ‘suggestion’, as in “I have a proposition for you.” In the present context, you should forget about this meaning. Think only of “proposition” as meaning ‘statement.’
<table>
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<tr>
<th>$P$</th>
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<th>$P \to Q$</th>
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That is, $A \to B$ is false only when $A$ is true but $B$ is false.

“$A \to B$” is variously read as: “$A$ implies $B$”, “if $A$ then $B$”, or “$A$ only if $B$”.

- In fact, $\land$ and $\neg$ are sufficient to define the other operations. That is, any expression involving $\to$ or $\lor$ can be translated into one with only $\land$ and $\neg$. 