Functions and Relations
Ling 106
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1. Sequences and Cartesian Products

A sequence of objects is a list of these objects in a particular order. (Recall that sets are unordered).

\(<a, b, c>\)
\(<b, a, c>\)
\(<c, b, a>\)

Finite sequences are called tuples. A sentence with k elements is a k-tuple. A 2-tuple is also called an (ordered) pair.

\(<0, 1>\)

If A and B are two sets, the Cartesian product of A and B, written as A×B, is the set of pairs wherein the first element is a member of A and the second element is a member of B.

Question 6.
Let K=\{a, b, c\} and L= \{1, 2\}

K×L = {\(<a, 1>\), \(<a, 2>\), \(<b, 1>\), \(<b, 2>\), \(<c, 1>\), \(<c, 2>\)}
L×K =
L×L =

Although each member of a Cartesian product is an ordered pair, the Cartesian product itself is an ordered set of them.

2. Relations

A relation is a set of pairs, e.g. mother of, less than, subset. A relation from A to B is a subset of the Cartesian product A×B.

Domain(R) = \{a | there is some b such that \(<a, b> \in R\}\)
Range(R) = \{b | there is some a such that \(<a, b> \in R\}\)

For example, let A = \{1, 2, 3, 4\} be the domain, and B = \{2, 3, 4\} be the range. We can visualize the two sets using Venn diagrams:
Let us now define a relation from A to B (from the domain to the range) as follows:
There exists a relation R between an element x in the domain A and an element y in the range B if x is less than y. We can represent this relation as follows:

Another way of representing this relation is to use ordered pairs:

The relation R from A to B = { <1, 2>, <1, 3>, <1, 4>, <2, 3>, <2, 4>, <3, 4> }
where A = {1, 2, 3, 4}, B = {2, 3, 4} and R is defined as:
xRy = { <x, y> where x < y and x ∈ A and y ∈ B }

R(a, b), Rab, or aRb: The relation R holds between objects a and b.

Note: A relation may relate one object in its domain to more than one object in its range.

The complement of a relation $R \subseteq A \times B$, written $R'$, contains all ordered pairs of the Cartesian product which are not members of the relation R. (Recall that $A \times B = \{ <a, c>, <a, d>, <a, e>, <b, c>, <b, d>, <b, e> \}$)
The inverse of a relation, written as $R^{-1}$, has as its members all the ordered pairs in $R$, with their first and second elements reversed.

$$(R')' = R$$
$$(R^{-1})^{-1} = R$$

Question 7.
Let $A=\{1, 2, 3\}$ and let $R \subseteq A \times A$ be $\{<3, 2>, <3, 1>, <2, 1>\}$, which is ‘greater than’ relation in $A$.
What is $R^{-1}$?
What is $R'$?

3. Functions

A relation $R$ from $A$ to $B$ is a function from $A$ to $B$ if and only if the following two condition holds:

- Condition A: each element in the domain is paired with just one element in the range (i.e., no elements may be paired with multiple elements in the range)
- Condition B: every element in the domain is paired with an element in the range (i.e., none are left out!)

In the above example, the relation $R$ from $A$ to $B$ is not a function because:

i) Element ‘1’ in $A$ is paired with elements ‘2’, ‘3’ and ‘4’ in $B$, violating Condition A. Likewise, element ‘2’ in $A$ is paired with elements ‘3’ and ‘4’ in $B$, also violating Condition A.
ii) Element ‘4’ in $A$ is not paired with any element in $B$, violating Condition B.

Here’s an example of a function:

Let $C = \{3, 4,\}$ and $D = \{2, 3, 4, 5\}$
Define the function $F$ as follows:

$\text{xFy} = \{ <x, y> \text{ where } x = y-1, \ x \in C \text{ and } y \in D \}$

![Diagram of function F]

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Verify that the function F satisfies both conditions A and B as defined above.

Question 8.
Let A={a, b, c} and B={1, 2, 3, 4}. Which of the following relations are functions?

a) P={<a, 1>, <b, 2>, <c, 3>}
b) Q={<a, 1>, <b, 2>}
a) R={<a, 2>, <a, 3>, <b, 4>}
b) S= {[a, 3>, <b, 2>, <c, 2>}

A function that is a subset of A×B is a function from A to B
A function in A×A is a function in A.

F: A -> B is read as “F is a function from A to B”.
F(a) =b is read as “F maps a to b”.

Functions from A to B are generally said to be into B.
Functions from A to B such that the range of the function equals B are called onto B.

A function F: A -> B is called a one-to-one function just in case no member of B is assigned to more than one member of A. Otherwise, we call it many-to-one function.

A function which is both one-to-one and onto is called a one-to-one correspondence. If a function F is a one-to-one correspondence, F⁻¹ is also a function.

Given F(a)=b, a is an argument and b is the value.
Unary function takes one argument. F(a)
Binary function takes two arguments. F(a, b)

Exercise 1.

The table above shows a (simplified) historical family tree of the Germanic group of languages, of which English is a member. Historical linguists assert that the Germanic languages all originated from a common proto-language called Common Germanic (or sometimes known as Proto-Germanic). Common Germanic later split into three distinct languages - North Germanic, West Germanic and East Germanic. The modern Germanic languages subsequently developed from these three earlier languages (though Gothic is now extinct).
For the questions below, the following definitions are important:

a) Let us call every language in the tree a **node**

b) A node X **dominates** another node Y if there is a descending path going from node X to node Y (e.g., Common Germanic or West Germanic dominate English)

c) A node X **immediately dominates** another node Y if X dominates Y and there is no intervening nodes in the path from X to Y (e.g., West Germanic immediately dominates English)

d) A node that dominates another node is called an **ancestor node** (e.g., Common Germanic and West Germanic are both ancestor nodes of English)

e) A node that immediately dominates another node is called a **mother node** (e.g., West Germanic is the mother node of English. Common Germanic is an ancestor node of English but it is not a mother node).

f) The node immediately dominated by a mother node is called the **child node** (for example, English is a child node of West Germanic, but not of Common Germanic).

g) A node X is a **sibling** of another node Y if X and Y share a common mother node (for example, English is a sibling of German, but not of Danish).

Let us define a group G which consists of all the Germanic languages found in the tree above:

\[ G = \{ \text{Common Germanic, North Germanic, West Germanic, East Germanic, Danish, Swedish, Norwegian, German, English, Dutch, Yiddish, Gothic} \} \]

For each of the relation below defined over the set G, write out the set of ordered pairs that are defined by that relation:

1. \( R1 = \) "is a mother of"

2. \( R2 = \) "is a child of"

3. \( R3 = \) "is a sibling of"

4. \( R4 = \) “is an ancestor of”

**Exercise 2.**

Let \( A = \{ b, c \} \) and \( B = \{ 2, 3 \} \)
Specify the following sets by listing their members.

(i) \( A \times B \)

(ii) \( B \times A \)

(iii) \( A \times A \)

**Exercise 3.**

Consider the following relation from A to B:

\[ R=\{<b, b>, <b, 2>, <c, 2>, <c, 3>\} \]

(i) Specify the domain and range of R
(ii) Specify the complimentary relation \( R' \) and the inverse \( R^{-1} \)

**Exercise 4.**

Let \( A=\{a, b, c, d\} \) and \( B=\{1, 2, 3\} \). Specify if the following relation is (a) not a function, (b) a function from A into B, (c) a function from A onto B, (a) one-to-one function?

(1) \( \{<a, 1>, <c, 2>, <b, 3>, <d, 1>\} \)
(2) \( \{<a, 1>, <b, 1>, <c, 3>, <d, 3>, <d, 2>\} \)
(3) \( \{<a, 2>, <a, 1>, <c, 3>, <d, 2>\} \)
(4) \( \{<a, 1>, <b, 2>, <c, 3>\} \)
(5) \( \{<a, 1>, <b, 1>, <c, 1>, <d, 1>\} \)