Solution to Optional Exercise
in Homework 7
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1 Exercise

Show that the following language is not regular.

\[ C = 001^n01^m, \] where \( n \geq 0, m \geq 0, \) and \( m \neq n. \)

Take \( p \) to be the pumping length given by the Pumping Lemma. We want to show that there is a string \( s \) at least \( p \)-long such that, no matter how we try to split it out into three pieces \( x, y \) and \( z \), it cannot be pumped according to the Pumping Lemma. In this way, we will show that the language \( C \) is not regular.

Of the following three following attempts, only the last one succeeds. You will need to understand why the others fail in order to understand why that one succeeds.

2 First Try.

- Choose \( s \) to be the string \( 001^{p-2}01^{p-1} \).

- According to the Pumping Lemma, if this language is regular, then \( s \) should be dividable into three pieces \( x, y \) and \( z \) such that:
  1. for each \( i \geq 0, xy^iz \in A, \)
  2. \( y \neq \epsilon, \) and
  3. \( |xy| \leq p. \)

- We want to show that there is no way to divide \( s \) that would be compliant with these three conditions. We consider these three cases:

  i. The string \( y \) contains only 0s. Just to see one example:

  \[
  \begin{array}{ccc}
  x & 0 & 0 \\
  y & 1^{p-2}01^{p-1} \\
  z & \\
  \end{array}
  \]

  But, if \( y \) contains exclusively 0s, then the pumped string \( xyyz \) has more 0s than the fixed ones allowed by the language \( C \). Hence, \( s \) cannot be pumped when its recursive part \( y \) consists only of 0s.

  ii. The string \( y \) contains 1s and 0s. Just to see one example:

  \[
  \begin{array}{ccc}
  x & 0 & 1 \\
  y & 01^{p-3}01^{p-1} \\
  z & \\
  \end{array}
  \]

  The pumped string \( xy^yz \) has more 0s than the ones allowed by \( C \) and furthermore they are interleaved with 1s in the wrong position. Hence, \( s \) cannot be pumped when its recursive part \( y \) consists of a mixture of 1s and 0s.
iii. The string $y$ contains only 1s. For example:

$$\begin{array}{ccc}
00 & 11 & 1^{p-1}01^{p-1} \\
\hline
x & y & z
\end{array}$$

Can $y$ be pumped and yield a string that still belongs to $C$? **Unfortunately, it can be pumped!** All the pumped strings belong to $C$:

$$xz = 001^{p-4}01^{p-1}$$

$$xyyz = 0011111^{p-4}01^{p-1} = 001^{p-1}$$

$$xyyzz = 001111111^{p-4}01^{p-1} = 001^{p-2}01^{p-1}$$

etc.

- We wanted to show that our string $s$ could not possibly be split in the way required by the Pumping Lemma. But we have found one way to split it that is compliant with it. Even if most splits would not satisfy the Pumping Lemma, there is one that does. This is not what we wanted.

- What have we shown so far? Nothing.

  We certainly have not shown that $C$ is regular. For that, we need to provide a FSA.

  We have to start all over again and try another string, until we find one that cannot possibly be divided in the way required by the Pumping Lemma.

3 **Second Try**

- Choose $s$ to be the string 0010.

- We want to show that there is no way to divide $s$ that would be compliant with these three conditions. Again, we consider these three cases:

  i. The string $y$ contains only 0s. In this case, the pumped string $xyyz$ has more 0s than the fixed ones allowed by $C$ and hence it does not belong to $C$.

  ii. The string $y$ contains 1s and 0s. Then, the pumped string $xyyz$ has more 0s than the ones allowed by $C$ and furthermore they are interleaved with 1s in the wrong position.

    Hence, the pumped string $xyyz$ does not belong to $C$.

  iii. The string $y$ contains only 1s. That is:

    $$\begin{array}{ccc}
    00 & 1 & 0 \\
    \hline
    x & y & z
    \end{array}$$

    In this case, the pumped-down string $xz$ contains the same number of 1s after the second 0 and after the third 0, namely no 1 at all. Hence, $xz$ does not belong to $C$, as desired.

- Hence, we have found a string, namely 0010, that cannot possibly be split in the way required by the Pumping Lemma.

- Are we done? Unfortunately, no. What we need is a string that is at least $p$-long that cannot be split in the relevant way. Is 0010 at least $p$-long? We cannot be sure. So, let us start all over again...
4 Third Try

- For any positive integer \( n \), the factorial of \( n \), written \( n! \), is the product of all positive integers from 1 to \( n \).
- \( 5! = 1 \times 2 \times 3 \times 4 \times 5 = 120 \)
- Choose \( s \) to be e.g. the string \( 001^p01^{p+p!} \).
- We want to show that there is no way to divide \( s \) that would be compliant with these three conditions. Again, we consider these three cases:
  
  i. The string \( y \) contains only 0s. The pumped string \( xyyz \) has too many 0s and hence it does not belong to \( C \).
  
  ii. The string \( y \) contains 1s and 0s. As before, the pumped string \( xyyz \) has too many 0s and in the wrong position. Thus, \( xyyz \) does not belong to \( C \).
  
  iii. The string \( y \) contains only 1s. Here we will consider several subcases:
  
  - The string \( y \) consists of exactly one 1. For example:
    
    \[
    \begin{array}{cccc}
    x & y & 1 & 01^{p+p!} \\
    \end{array}
    \]
    
    Now, if you pump \( y \) \( p!+1 \) times, you will obtain the string \( 001^{p+p!}01^{p+p!} \). Since it has the same number of 1s in both positions, this string does not belong to \( C \).
  
  - The string \( y \) consists of exactly two 1s. For example:
    
    \[
    \begin{array}{cccc}
    x & y & 11 & 01^{p+p!} \\
    \end{array}
    \]
    
    Now, if you pump \( y \) \( (p!/2)+1 \) times, you will obtain the string \( 001^{p+p!}01^{p+p!} \), which does not belong to \( C \).
  
  - The string \( y \) consists of exactly three 1s. For example:
    
    \[
    \begin{array}{cccc}
    x & y & 111 & 01^{p+p!} \\
    \end{array}
    \]
    
    Now, if you pump \( y \) \( (p!/3)+1 \) times, you will obtain the string \( 001^{p+p!}01^{p+p!} \), which does not belong to \( C \).
  
  - The string \( y \) consists of exactly four 1s. For example:
    
    \[
    \begin{array}{cccc}
    x & y & 1111 & 01^{p+p!} \\
    \end{array}
    \]
    
    Now, if you pump \( y \) \( (p!/4)+1 \) times, you will obtain the string \( 001^{p+p!}01^{p+p!} \), which does not belong to \( C \).
  
  - And so on and so forth until \( y \) contains as possible (a number of \( p-2 \) 1s for the current string \( s \), since the length of \( xy \) must be at most \( p \)).
  
  - In sum, when \( y \) contains only 1s, it cannot possibly be divided in the way required the Pumping Lemma.

- The key point was to find a number \( p+p! \) of digits that can be reached by starting with \( p \)-many digits and repeating any chunk of them in steps of one digit, two digits, three digits, etc. Since \( p! \) is divisible by any positive integer between 1 and \( p \), it can be reached by repeating chunks that contain between one and \( p \)-many digits. Now we are done.