1 Set Theory

1.1 What is a set?

- A set is a collection of objects. It can be finite or infinite.
  \[ A = \{a, b, c\} \]
  \[ N = \{1, 2, 3, \ldots\} \]

- An object is an element of a set \( A \) if that object is a member of the collection \( A \).
  Notation: “\( \in \)” reads as “is an element of” or “belong to”.
  \[ a \in A \]
  \[ 2 \in N \]

A set can have another set as a member.
Let \( B = \{a, b, c, \{d, e\}\} \), then \( \{d, e\} \in B \)

- A set with only one member is called a **singleton**.

- A set with no members is called the **empty set** or **null set**, \( \emptyset \).

1.2 Specification of sets

- List notation
  A set consists of the objects named, not of the names themselves.
  \[ B = \{\text{The Amazon River, George Washington, 3}\} \]
  \[ C = \{\text{The Amazon River, ‘George Washington’, 3}\} \]
  A set is unordered.

  Writing the name of a member more than once does not change its membership status. For a given object, either it is a member of a given set or it is not.
  \[ \{a, b, c, e, e, e\} \]
  \[ \{a, b, c, e\} \]

- Predicate notation
  A better way to describe an infinite set is to indicate a property the members of the set share.
  \[ \{x | x \text{ is an even number greater than 3}\} \] is read as
  “the set of all \( x \) such that \( x \) is an even number greater than 3.”

  ‘\( x \)’ is a variable.

**Russell’s Paradox:**
Let \( U = \{x | x \notin x\} \). Is \( U \) a member of itself? (i) if \( U \) is not a member of itself, then it satisfies the property defined as \( x \notin x \), therefore it must be a member of \( U \).
(ii) if \( U \) is a member of itself, then it does not satisfy the property defined as \( x \notin x \), hence it is
not a member of $U$.

$\Rightarrow$ Logical Paradox!!!

One solution: Type Theory.

- Recursive rules
  A rule for generating elements ‘recursively’ from a finite basis.
  a) $4 \in E$
  b) if $x \in E$, then $x + 2 \in E$
  c) Nothing else belongs to $E$.

Question: Give a list notation for the above recursive rules.

1.3 Set-theoretic identity and cardinality

- Two sets are identical if and only if they have exactly the same members.
  \[ \{1, 2, 3, 4\} = \{x \mid x \text{ is a positive integer less than 5}\} \]

- The number of members in a set $A$ is called the cardinality of $A$, written $|A|$.

Let $A = \{1, 3, 5, a, b\}$. $|A| =$

1.4 Subsets

- A set $A$ is a subset of a set $B$ if all the elements of $A$ are also in $B$ (notation: $A \subseteq B$).

- Proper subset (notation: $A \subset B$).

- $A \not\subseteq B$ means that $A$ is not a subset of $B$.

Question: Fill in the blank with either $\subseteq$ or $\not\subseteq$.

a) \{a, b, c\} \quad \{s, b, a, e, g, i, c\}
b) \{a, b, j\} \quad \{s, b, a, e, g, i, c\}
c) \emptyset \quad \{a\}
d) \{a, \{a\}\} \quad \{a, b, \{a\}\}
e) \{\{a\}\} \quad \{a\}
f) \{a\} \quad \{\{a\}\}
g) \{\emptyset\} \quad \{a\}
h) $A$ \quad A

Question: Are the following statements true or false?

Let $A = \{b, \{c\}\}$.

a) \(c \in A\)
b) \(\{c\} \in A\)
c) \(\{b\} \subseteq A\)
d) \(\{c\} \subseteq A\)
e) \(\{\{c\}\} \subseteq A\)
f) \(\{b\} \not\subseteq A\)
g) \(\{b, \{c\}\} \subset A\)
h) \(\{\{b, \{c\}\}\} \subseteq A\)

- A set $A$ is a subset of itself. $A \subseteq A$.  

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• If $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$.

1.5 Power sets

• The power set of $A$, $\mathcal{P}(A)$, is the set whose members are all the subsets of $A$.
  Let $A = \{a, b\}$.
  $\mathcal{P}(A) = \{\{a\}, \{b\}, \{a, b\}, \emptyset\}$

• $|\mathcal{P}(A)| = 2^n$

Question: Let $B = \{a, b, c\}$. What is $\mathcal{P}(B)$?

1.6 Set-theoretic operations

• $A \cap B$: The intersection of two sets $A$ and $B$ is the set containing all and only the objects that are elements of both $A$ and $B$.
  $A \cap B \cap C = \bigcap \{A, B, C\}$

• $A \cup B$: The union of two sets $A$ and $B$ is the set containing all and only the objects that are elements of $A$, or of $B$, or of both $A$ and $B$.
  $A \cup B \cup C = \bigcup \{A, B, C\}$

• $A - B$: The difference between two sets $A$ and $B$ subtracts from $A$ all objects which are in $B$.

• $A^c$: The complement of a set $A$ is the set of all the individuals in the universe of discourse except for the elements of $A$ (i.e., $U - A$).

Question: Let $K = \{a, b\}$, $L = \{c, d\}$, and $M = \{b, d\}$.
  a) $K \cup L = \{a, b, c, d\}$
  b) $K \cup M = \{a, b, c, d\}$
  c) $(K \cup L) \cup M = \{a, b, c, d\}$
  d) $L \cup \emptyset = \emptyset$
  e) $K \cap L = \emptyset$
  f) $L \cap M = \emptyset$
  g) $K \cap K = K$
  h) $K \cap \emptyset = \emptyset$
  i) $K \cap (L \cap M) = \emptyset$
  j) $K \cap (L \cup M) = \{a\}$
  k) $K - M = \{a\}$
  m) $L - M = \emptyset$
  n) $M - L = \{b\}$
  o) $K - \emptyset = K$
  p) $\emptyset - K = \emptyset$

1.7 Set-theoretic equalities

see pg. 18 of ch. 1, Partee, ter Mullen and Wall.
Set-theoretic equalities can be used to simplify a complex set-theoretic expression, or to prove the truth of other statements about sets.
• Simplify the expression $(A \cup B) \cup (B \cap C)'$.

1. $(A \cup B) \cup (B \cap C)'$
2. $(A \cup B) \cup (B' \cup C')$ DeMorgan
3. $A \cup (B \cup (B' \cup C'))$ Associative
4. $A \cup ((B \cup B') \cup C')$ Associative
5. $A \cup (U \cup C')$ Complement
6. $A \cup (C' \cup U)$ Commutative
7. $A \cup U$ Identity
8. $U$ Identity

**Question:** Show that $(A \cap B) \cap (A \cap C)' = A \cap (B - C)$.

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**Question:** Show that $X \cap Y \subseteq X \cup Y$. (hint: Use Consistency Principle.)

2 Ordered pairs and Cartesian products

• A sequence of objects is a list of these objects in some order. (cf., Recall that a set is unordered.)
  $< a, b, c >, < 7, 21, 57 >, < 1, 2, 3, ... >$

  Finite sequences are called **tuples**. A sequence with $k$ elements is a $k$-**tuple**.

  A 2-tuple is also called an (ordered) **pair**.

  $< a, b >$

  • If $A$ and $B$ are two sets, the **Cartesian product** of $A$ and $B$, written as $A \times B$, is the set containing all pairs wherein the first element is a member of $A$ and the second element is a member of $B$.

    **Question:** Let $K = \{a, b, c\}$ and $L = \{1, 2\}$.

    $K \times L = \{< a, 1 >, < a, 2 >, < b, 1 >, < b, 2 >, < c, 1 >, < c, 2 >\}$

    $L \times K =$

    $L \times L =$

  • Although each member of a Cartesian product is an ordered pair, the Cartesian product itself is an unordered set of them.
3 Relations

- A relation is a set of pairs. e.g., mother of, kiss, subset.
  A relation from $A$ to $B$ is a subset of the Cartesian product $A \times B$.
  $R(a, b)$, $Rab$, or $aRb$: The relation $R$ holds between objects $a$ and $b$.
  $R \subseteq A \times B$: A relation between objects from two sets $A$ and $B$. A relation from $A$ to $B$.
  $R \subseteq A \times A$: A relation holding of objects from a single set $A$ is called a relation in $A$.

- Domain($R$) = $\{a\}$ if there is some $b$ such that $<a, b> \in R$
  Range($R$) = $\{b\}$ if there is some $a$ such that $<a, b> \in R$
  Let $A = \{a, b\}$ and $B = \{c, d, e\}$. $R = \{<a, d>, <a, e>, <b, c>\}$.
  Domain($R$) = $\{a, b\}$
  Range($R$) = $\{c, d, e\}$
  Note: A relation may relate one object in its domain to more than one object in its range.

- A relation $R$ is an equivalence relation if $R$ satisfies the following three conditions:
  $R$ is reflexive, i.e., for every $x \in \text{Domain}(R)$, $R(x, x)$.
  $R$ is symmetric, i.e., for every $x$ and $y \in \text{Domain}(R)$, $R(x, y)$ if and only if $R(y, x)$.
  $R$ is transitive, i.e., for every $x$, $y$ and $z \in \text{Domain}(R)$, $R(x, y)$ and $R(y, z)$ implies $R(x, z)$.

- The complement of a relation $R \subseteq A \times B$, written $R'$, contains all ordered pairs of the Cartesian product which are not members of the relation $R$.
  The inverse of a relation, written as $R^{-1}$, has as its members all the ordered pairs in $R$, with their first and second elements reversed.

$$(R')' = R$$
$$(R^{-1})^{-1} = R$$

If $R \subseteq A \times B$, the $R^{-1} \subseteq B \times A$, but $R' \subseteq A \times B$.

Question: Let $A = \{1, 2, 3\}$ and let $R \subseteq A \times A$ be $<3, 2>, <3, 1>, <2, 1>$, which is ‘greater than’ relation in $A$.
  What is $R'$?
  What is $R^{-1}$?

4 Functions

- A relation $R$ from $A$ to $B$ is a function from $A$ to $B$ if and only if:
  a) Each element in the domain is paired with just one element in the range.
  b) The domain of $R$ is equal to $A$ (except for partial functions).

Question: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Which of the following relations are functions?
  a) $P = \{<a, 1>, <b, 2>, <c, 3>\}$
  b) $Q = \{<a, 1>, <b, 2>\}$
  c) $R = \{<a, 2>, <a, 3>, <b, 4>\}$
  d) $S = \{<a, 3>, <b, 2>, <c, 2>\}$
• A function that is a subset of $A \times B$ is a function from $A$ to $B$.
  A function in $A \times A$ is a function in $A$.

• $F : A \rightarrow B$ is read as “$F$ is function from $A$ to $B$”.

• $F(a) = b$ is read as “$F$ maps $a$ to $b$”.

• Given $F(a) = b$, $a$ is an argument and $b$ is the value.

• Functions from $A$ to $B$ are generally said to be into $B$.
  Functions from $A$ to $B$ such that the range of the function equals $B$ are called onto $B$.

• A function $F : A \rightarrow B$ is called a one-to-one function just in case no member of $B$ is assigned
  to more than one member of $A$.
  Otherwise, we will call them many-to-one function.

• A function which is both one-to-one and onto is called a one-to-one correspondence.
  If a function $F$ is a one-to-one correspondence, $F^{-1}$ is also a function.

• A function with $k$ arguments is called a $k$-ary function, and $k$ is called the arity of the
  function.
  Unary function takes one argument. $F(a)$.
  Binary function takes two argument. $F(a, b)$.

• Infix notation: e.g., $a + b$.
  Prefix notation: e.g., $+(a, b)$.

• A predicate or property is a function whose range is \{True, False\}.
  $\text{even}(2) = \text{True}$, $\text{even}(3) = \text{False}$.
  take-106, male, freshman.