Set Theory

LING 255

January 24, 2008

In a moment, set theory. But first:

1. From Tuesday...

I suggested that ⟦**snow is white**⟧ = TRUE (as long as snow is white). Ben expressed qualms on, I think, two points: first, is TRUE (or FALSE) really an object in the world, the way, say, Will Russell is; second, isn't its truth or falsity situation-dependent?

Both points are valid. The best answer I can give is in terms of Frege's *Sinn und Bedeutung*, usually translated "sense and reference". In essence, the "sense" of a term is its reference across situations; the "reference" of a term is what its sense is in a particular situation.

For instance:

Term	Reference	Sense
George W. Bush	George W. Bush	George W. Bush (in any given situation, **George W. Bush** refers to the same particular individual)
the president of the United States	George W. Bush (i.e., he's the individual picked out by this phrase)	in the current situation, George W. Bush ten years ago, Bill Clinton two years from now, someone else etc.
George W. Bush is the president of the United States	TRUE	in the current situation, TRUE ten years ago, FALSE two years from now, FALSE

For a sentence such as **snow is white** or **George W. Bush is the president of the United States**, Ben is right that we can think of it as meaning "those situations in which it's true". That's the sense of the sentence. But for that to be well-defined, we need a reference for the sentence to have in any situation, and that reference needs to be, roughly, "it's true" or "it's false". That's what we'll be using here.

2. SETS

Definition: A **SET** is simply a collection of objects.
Definition: An object in a set is called an **ELEMENT**. "*x* is an element of *S*" (or "*x* is a member of *S*") is written $x \in S$.
Definition: The number of elements in a set is called the **CARDINALITY** of the set. "The cardinality of *S*" is written $|S|$.
Definition: A is **IDENTICAL TO** B, written A = B, iff they have exactly the same members.

Some sample sets:

- *A* = {Brown, Columbia, Cornell, Dartmouth, Harvard, Penn, Princeton, Yale}
- *B* = {1, 2, 3, 4, 5, 6, 7, 8, ...}
- *C* = {George W. Bush, my laptop, 5, the planet Mars}

- Set membership:
  $5 \in B$. $5 \in C$. $5 \notin A$.[1]

- Cardinality:
  $|A|$ = ______? $|B|$ = ______? $|C|$ = ______?

Notes: Sets are unordered, and something can only be an element of a set once. Repeating it when writing out the set has no effect.

- {5, George W. Bush, the planet Mars, my laptop} = *C*
- {5, George W. Bush, the planet Mars, George W. Bush, 5, my laptop, 5, 5, 5, 5} = *C*

Elements of sets are objects, not the names of objects (though linguistic strings such as names can be elements of sets). Consequently, it doesn't matter how you name the elements of a set.

- {John Lennon, George Harrison, Paul McCartney, Ringo Starr}
  ≠ {**John Lennon**, **George Harrison**, **Paul McCartney**, **Ringo Starr**}

  (the first is a set of four people; the second is a set of four names)

- {the man holding the office of President of the United States in July 2003, this Dell Inspiron, the positive square root of 25, the fourth most distant planet from Sol} = *C*

Sets can be elements of other sets.

- *Q* = {set *A*, set *B*, set *C*}
- *E* = {1, 2, 3, 4, {1, 2}, {3, 4}, {1, 2, 3, 4}} **Question: $|E|$ = _____?**

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[1] Standard notation: a backslash through a symbol means "not". $\notin$ means "is not a member of"; ≠ means "is not equal to".

A few more examples:

- $S = \{1, 2, 3, 4, 5, 6, 7\}$ $|S| = 7$
- $S = \{1, 2, 3, 4, 5, \{6, 7\}\}$ $|S| = 6$ (because {6, 7} is a single element)
- $S = \{1, 2, \{1, 2\}, \{2\}, \{3\}\}$ $|S| = 5$

This last example may be particularly illuminating: note that {1, 2} is a single element distinct from either 1 or 2; and that {2} is also an element distinct from 2.

2.1. Some special sets

- The set of everything being considered is called **THE UNIVERSE** and is written *U*. (This will vary from context to context.)
- Any set with one member is called a **SINGLETON SET**. e.g., {John Lennon}, {5}.
- The set with no members at all is called the **EMPTY SET** (or **NULL SET**) and written ∅. ($|\varnothing| = 0$.)
- Because we'll be working with the ideas of "true" and "false", we'll set up a set of *truth values*: D_t = {TRUE, FALSE}. (Because it doesn't matter how we name things, we could also use {T, F}, or {0, 1}, or {♣, ♠} or anything else.)

2.2. How to specify a set

2.2.1. List Notation

The notation used above. Simply write out a list of the elements of the set.

Advantages
- It's immediately clear what's in the set.
- It works for pretty much any set.

Disadvantages
- The larger the cardinality of the set, the harder it is to write out the full list.
- Once the set is infinite, literally listing the elements becomes impossible. It can still be done using ellipses, but it may be ambiguous or unclear:

$F = \{1, 3, 5, 7, ...\}$
$P = \{6, 28, 496, ...\}$

2.2.2. Predicate Notation

Describe the members of the set, rather than naming each one. Frequently used notation: "the set of all x for which it's true that [condition]" is written $\{x \mid [\text{condition}]\}$. Examples:

- A = the set of Ivy League schools, *or*
  {the Ivy League schools}, *or*
  {x such that x is an Ivy League school}, *or*
  {x | x is an Ivy League school}

- B = {x | x is a positive integer}

Advantages
- Concise and unambiguous, both of which are quite handy for infinite sets

Disadvantages
- Doesn't work for every set (good luck expressing C this way}
- Tears holes in the space-time continuum...

> **<u>Russell's Paradox</u>**
>
> Let S be the set of sets which do not contain themselves, i.e. $S = \{x \mid x \notin x\}$.
> Question: is S a member of S?
>
> (i) If S is a not a member of S, then it matches the criterion for inclusion in S, and therefore it must be a member of S.
> (ii) If S is a member of S, then that's because it matches the criterion for inclusion in S, i.e. because it's not a member of itself, i.e., S is not a member of S.
>
> Therefore, S is neither a member of S, nor not a member of S.

Moral of the story: just because you can write it down doesn't mean you *should*. Or, a little more formally: just because you can write it in predicate notation doesn't make it a well-defined set.

- One solution: everything in a set should be the same "type", e.g.
  ◦ Integers; people; sets of integers; sets of sets of integers...
  This will happen to be a feature of what we do, though we won't require it.

2.3. Things to do with sets

What can we do with sets? We can talk about their elements and cardinality, of course. We can also talk about relations between sets:

Definition: A is a **SUBSET** of B ($A \subseteq B$) iff every element of A is an element of B.
A is a **PROPER SUBSET** of B ($A \subset B$) iff $A \subseteq B$ and $A \neq B$.[2]

- {George W. Bush, the planet Mars} $\subseteq C$
- $\{x \mid x$ is evenly divisible by 6$\} \subseteq \{x \mid x$ is evenly divisible by 2$\}$
- $\{x \mid x$ is evenly divisible by 6$\} \subset \{x \mid x$ is evenly divisible by 2$\}$
- $\{x \mid x$ is evenly divisible by 6$\} \not\subseteq \{x \mid x$ is evenly divisible by 4$\}$

And we have operations on sets:

Definition: The **INTERSECTION** of A and B ($A \cap B$) is the set that contains all and only those elements that are in *both* A *and* B.

Definition: The **UNION** of A and B ($A \cup B$) is the set that contains all and only those elements that are in A *or* B (or both).[3]

Definition: The **DIFFERENCE** of A and B ($A - B$) is the set of all individuals that are in A and not in B.

Definition: The **COMPLEMENT** of A (A') is the set of all individuals that are not in A, with respect to some "universe of discourse" U: that is, $U - A$.

- $\{1, 2, 3, 4, 5, 6\} \cap \{2, 4, 6, 8, 10, 12\} = \{2, 4, 6\}$
- $\{1, 2, 3, 4, 5, 6\} \cup \{2, 4, 6, 8, 10, 12\} = \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$

- $\{x \mid x$ is an Ivy League school$\} \cap \{x \mid x$ is in the state of New York$\}$ = ________
- $\{x \mid x$ is an Ivy League school$\} \cup \{x \mid x$ is in the state of New York$\}$ = ________

- $\{x \mid x$ is a square number$\} \cap \{x \mid x$ is a prime number$\}$ = ________

- With respect to the set of integers: $\{x \mid x$ is an odd integer$\}' = \{x \mid x$ is an even integer$\}$
- With respect to the set of prime numbers: $\{x \mid x$ is an odd integer$\}'$ = _______

- $\{1, 2, 3, 4, 5, 6\} - \{2, 4, 6, 8, 10, 12\} = \{1, 3, 5\}$

Definition: The **POWER SET** of A, $\wp(A)$, is the set of all subsets of A.
- If $H = \{a, b, c\}$, then $\wp(H) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
- $|\wp(S)| = 2^{|S|}$

[2] Also: A is a (**PROPER**) **SUPERSET** of B iff B is a (proper) subset of A, written $A \supseteq B$, $A \supset B$. But this is rare.
[3] Additional notation: $A \cap B \cap C$ is often written as $\bigcap\{A, B, C\}$, and $A \cup B \cup C$ as $\bigcup\{A, B, C\}$.

3. TUPLES

Definition: A **SEQUENCE** is a list of objects in a particular order.

- <*a*, *b*, *c*>; <George W. Bush, my laptop, 5, the planet Mars>; <1, 2, 3, 4, ...>
- {*a*, *b*, *c*} = {*c*, *b*, *a*}, but <*a*, *b*, *c*> $\neq$ <*c*, *b*, *a*>
- {*a*, *b*} = {*a*, *b*, *b*}, but <*a*, *b*> $\neq$ <*a*, *b*, *b*>

Finite sequences are also called **TUPLES**; a sequence with *n* elements is an *n*-tuple. (A 2-tuple is usually called an “ordered pair”.)

Definition: The **CARTESIAN PRODUCT** of two sets A and B (A $\times$ B) is the set of ordered pairs: {<*x*, *y*> | *x* $\in$ A and *y* $\in$ B}

- If *H* = {*a*, *b*, *c*} and *J* = D_t, then
  - *H* $\times$ *J* = {<*a*, F>, <*b*, F>, <*c*, F>, <*a*, T>, <*b*, T>, <*c*, T>}
  - *J* $\times$ *J* = ____

4. RELATIONS

Definition: A **RELATION** is a set of pairs.
The **DOMAIN** of a relation R is {*x* | there is some *y* such that <*x*, *y*> $\in$ R}.
The **RANGE** of a relation R is {*y* | there is some *x* such that <*x*, *y*> $\in$ R}.

Specifically: If R $\subseteq$ A $\times$ B, then R is a *relation from A to B*.
If R $\subseteq$ A $\times$ A, then R is a *relation in A*.
If <*x*, *y*> $\in$ R, also written R(*x*, *y*), R*xy*, or *x*R*y*, then R *holds* between *x* and *y*.

For example:

- Let *S* = {Bart, Homer, Grandpa, Lisa, Maggie, Marge}.
  Define the “is the father of” relation *F* in *S*: *F* $\subseteq$ *S* $\times$ *S*.

- *F* = {<Grandpa, Homer>, <Homer, Bart>, <Homer, Lisa>, <Homer, Maggie>}
- The domain of *F* = {Grandpa, Homer}
- The range of *F* = {Homer, Bart, Lisa, Maggie}

Definition: The **COMPLEMENT** of a relation R $\subseteq$ A $\times$ B, written R′, is the set of pairs in A $\times$ B that are not in R, i.e. {<*x*, *y*> | <*x*, *y*> $\notin$ R}

- *F*′ = {<Homer, Marge>, <Homer, Grandpa>, <Bart, Lisa>, ...}

Definition: The **INVERSE** of a relation R $\subseteq$ A $\times$ B, written R^{-1}, is the set of pairs in R with their elements reversed, i.e. {<*y*, *x*> | <*x*, *y*> $\in$ R}

- F^{-1} = {<Homer, Grandpa>, <Bart, Homer>, <Lisa, Homer>, <Maggie, Homer>}

Note that if $R \subseteq A \times B$, then $R' \subseteq A \times B$ and $R^{-1} \subseteq B \times A$.

5. FUNCTIONS

Definition: *F* is a **FUNCTION** from A to B, written $F : A \rightarrow B$, if

F is a relation from A to B such that:
(a) each element in the domain of F maps to only one element in the range, and
(b) domain(F) = A

Note: if domain(F) $\subset$ A, then F is called a *partial function*. In general, "function" by itself is used for complete functions only, though we may occasionally see partial functions.

For example: If $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$, then which of the following relations from A to B are functions?

- *P* = {<*a*, 1>, <*b*, 2>, <*c*, 3>}
- *Q* = {<*a*, 1>, <*b*, 2>}
- *R* = {<*a*, 2>, <*a*, 3>, <*b*, 4>, <*c*, 1>}
- *S* = {<*a*, 3>, <*b*, 2>, <*c*, 2>}

Further notation and definitions:

- If <*x*, *y*> $\in$ *F*, then $F(x) = y$, read "*F* maps *x* to *y*".
  In $F(x) = y$: *x* is the **ARGUMENT**, *y* is the **VALUE**.

- If each element in the range of *F* is mapped to by only one element in the domain—i.e., the converse of (a) in the definition—then *F* is **ONE-TO-ONE**. (If not, *F* is MANY-TO-ONE.)
- If range(*F*) = B—i.e., the converse of (b)—then *F* is **ONTO** (or a function "onto B"). (If not, *F* is INTO or a function "into *B*".)
- If *F* is one-to-one and onto, *F* is called a **ONE-TO-ONE CORRESPONDENCE**. (Note that in this case F^{-1} is also a function.)

A function that's neither one-to-one nor onto:
A = {*a*, *b*, *c*}, *B* = {0, 1, 2}, *F* : *A* → *B* = {<*a*, 0>, <*b*, 1>, <*c*, 0>}

A function that's one-to-one but not onto:
A = {*a*, *b*, *c*}, *B* = {0, 1, 2, 3}, *F* : *A* → *B* = {<*a*, 0>, <*b*, 1>, <*c*, 2>}

A function that's onto but not one-to-one:
A = {*a*, *b*, *c*}, *B* = {0, 1}, *F* : *A* → *B* = {<*a*, 0>, <*b*, 1>, <*c*, 0>}

A function that's both one-to-one and onto:
A = {*a*, *b*, *c*}, *B* = {0, 1, 2}, *F* : *A* → *B* = {<*a*, 0>, <*b*, 1>, <*c*, 2>}
(note that *F* looks a lot like the not-onto function above; it matters what set the function is a function into.)

6. Functions and Sets

Suppose that U = {a, b, c, d, e, f, g, h, i, j}, and S = {a, b, c, d, e}.

- a $\in$ S: True. b $\in$ S: True. f $\in$ S: False. (etc. etc.)

Rather than giving this kind of list, we can write a function $f{:}U \rightarrow D_t$ to represent it:

$$f = \begin{bmatrix} \mathtt{a} \rightarrow \text{TRUE} \\ \mathtt{b} \rightarrow \text{TRUE} \\ \mathtt{c} \rightarrow \text{TRUE} \\ \mathtt{d} \rightarrow \text{TRUE} \\ \mathtt{e} \rightarrow \text{TRUE} \\ \mathtt{f} \rightarrow \text{FALSE} \\ \mathtt{g} \rightarrow \text{FALSE} \\ \mathtt{h} \rightarrow \text{FALSE} \\ \mathtt{i} \rightarrow \text{FALSE} \\ \mathtt{j} \rightarrow \text{FALSE} \end{bmatrix}$$

This is the **characteristic function** of S: the function such that

$$\text{for all } x \in U,\ f(x) = \begin{cases} \text{TRUE} & \text{if } x \in \text{S} \\ \text{FALSE} & \text{otherwise} \end{cases}.$$

This turns a set into a function. We can turn a function *f* into the set "characterized by" *f*:

S = {the individuals *x* such that *f* maps *x* to TRUE} *or*
S = {$x \in U \mid f(x)$ = TRUE}

i.e., f(a) = TRUE is equivalent to a $\in$ S.

- Quick moral: functions are different from sets. But we can sometimes talk about them interchangeably.

6.1. What about algebra?

The kinds of functions that are more familiar....$g{:}\mathbb{N} \rightarrow \mathbb{N}$:[4]

$$g = \begin{bmatrix} 1 & \rightarrow & 2 \\ 2 & \rightarrow & 3 \\ 3 & \rightarrow & 4 \\ 4 & \rightarrow & 5 \\ & \ldots & \end{bmatrix}$$

i.e., $g(x) = x + 1$, i.e. the function from natural numbers to natural numbers such that it maps each natural number to its successor.

There's not really a set characterized by this function. Generally: a set containing elements of S can be expressed as a function $f{:}S \rightarrow D_t$. And a function can be expressed as a set if the range of the function is D_t.

But: since functions (for us) are ordered pairs, we should really have...

g = {<1,2>, <2, 3>, <3, 4>, <4, 5>, ...} $\equiv$
g(<1,2>) = TRUE, g(<1,3>) = FALSE, g(<2,2>) = TRUE, ...

i.e. $g : (\mathbb{N} \times \mathbb{N}) \rightarrow D_t$.

[4] $\mathbb{N}$ = the set of positive integers = {1, 2, 3, 4, ...}