Formal Reasoning and Analysis

LING 106, Jan. 14, 2009

1. A FORMAL SYSTEM: SHAPES

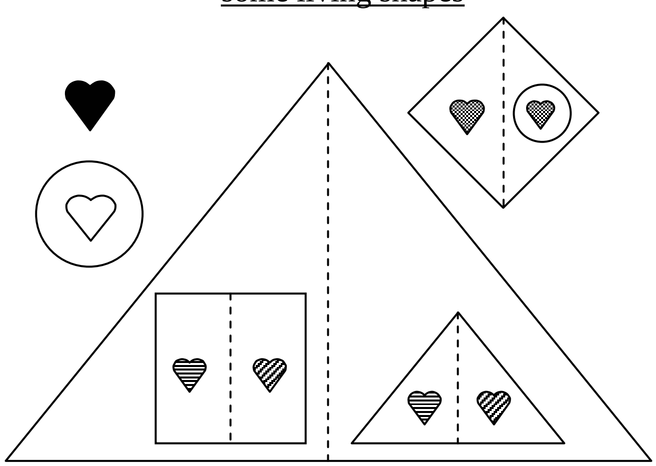
> *Chesterton was easily excited. He once wrote about seeing "evil shapes" in the pattern of a Turkish carpet — an odd idea that turns up in C. S. Lewis's Narnia too, where the Witch kills Aslan with a knife of "a strange and evil shape." What is an* evil shape, *I wonder? Could a triangle be evil, for example? Are some kinds of triangles decent and God-fearing, whereas others are treacherous...? And could you tell that from the* shape*?*
>
> *—Philip Pullman, "The Republic of Heaven"*

1.1. Living and Dead

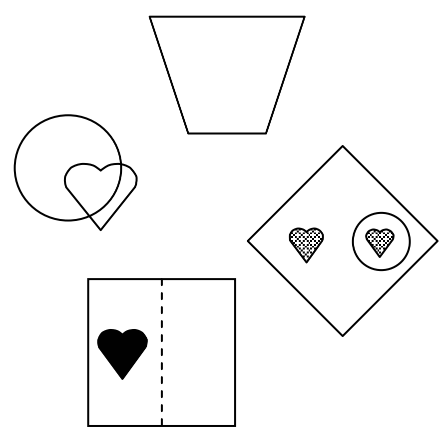
Suppose we classify shapes into "living" and "dead" by the following system.

- A shape is "living" if:
  - It is a heart (black, white, or colored/shaded)
  - It is a circle that contains a living shape
  - It is a bisected diamond, triangle, or square that contains a living shape in each half.
- Otherwise, the shape is "dead".

some living shapes



some dead shapes



1.2. Good and Evil

The following are the rules that determine whether a shape is “good” or “evil”.

- **Heart Rule (HR)**
  Black hearts are good. White hearts are evil. Other hearts are unknown.

- **Circle Rule (CR)**
  A circle containing a good shape is evil.
  A circle containing an evil shape is good.

- **Double Circle Rule (DCR)**
  If a shape is good, then a circle containing a circle containing the shape is good.
  If a circle containing a circle containing a shape is good, then the shape is good.

- **Square Rule (SR)**
  If two shapes are good, then a square containing the two shapes is good.
  Otherwise, the square is evil.

  If a square is good, then both of the shapes it contains are good.

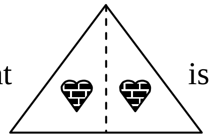
- **Diamond Rule (DR)**
  If two shapes are evil, then a diamond containing the two shapes is evil.
  Otherwise, the diamond is good.

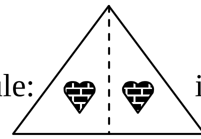
- **Triangle Rule (TR)**
  A triangle is evil if its left shape is good, and its right shape is evil.
  Otherwise, the triangle is good.

- **Triangle Formation Rule (TFR)**
  If you assume a shape *x* is good, and from that you can show that shape *y* is good, then a triangle with *x* on the left and *y* on the right is a good shape.

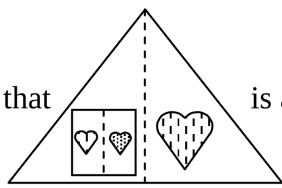
- **Triangle Piece Rule (TPR)**
  If a triangle is good, and the left shape in the triangle is good, then the right shape must also be good.

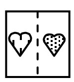
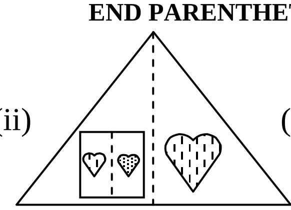
1.3. Proofs

(1) Prove that  is a good shape.

(i) **START PARENTHETICAL**
  (a) Assume:  is a good shape.
  (b) From (a):  is a good shape.
  END PARENTHETICAL
(ii) From (i) + the Triangle Formation Rule:  is a good shape

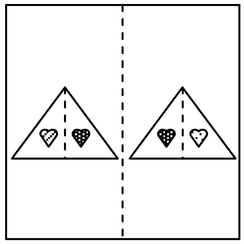
The key elements of a proof:
- Each line contains a good shape.
- Each line contains an explanation of why the shape is good.
- Parentheticals are clearly marked and set apart. (In the in-class slides, I used actual parentheses. Whatever works.)
  - Assumptions are only allowed at the start of a parenthetical.
  - Note that shapes that are good inside a parenthetical aren't necessarily *actually* good shapes! Don't use them as good shapes once the parenthetical is finished!

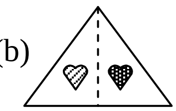
(2) Prove that  is a good shape.

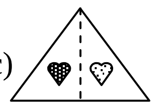
(i) **START PARENTHETICAL**
  (a)  [assumption]
  (b)  (a) + SR
  END PARENTHETICAL
(ii)  (i) + TFR

(3) Prove that the shape at the bottom of the page is a good shape.

(1) **START PARENTHETICAL A**

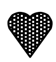
(a)  [assumption]

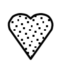
(b)  [(a) + SR]

(c)  [(a) + SR]

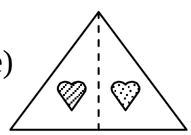
(d) **START PARENTHETICAL B**

(i)  [assumption]

(ii)  [TPR + (b) + (i)]

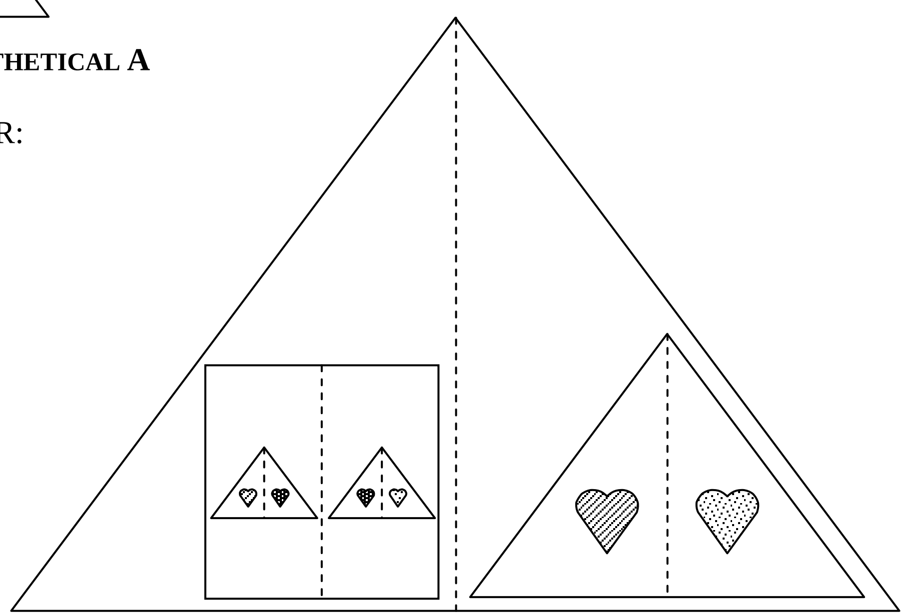
(iii)  [TPR + (c) + (ii)]

END PARENTHETICAL B

(e)  [(d) + TFR]

END PARENTHETICAL A

(2) By (1) + TFR:



2. SETS

Definition: A **SET** is simply a collection of objects.

Some examples:

- A = {Brown, Columbia, Cornell, Dartmouth, Harvard, Penn, Princeton, Yale}
- $B = \{1, 2, 3, 4, 5, 6, 7, 8, ...\}$
- C = {George W. Bush, my laptop, 5, the planet Mars}

A set with one member is called a **SINGLETON SET**. There is also one special set, the set with no members at all, which is called the **EMPTY SET** (or **NULL SET**) and written $\varnothing$.

2.1. Some properties of sets

Definition: An object in a set is called an **ELEMENT**. "x is an element of S" is written $x \in S$.

- $5 \in B$. $5 \in C$. $5 \notin A$.

Definition: The number of elements in a set is called the **CARDINALITY** of the set. "The cardinality of S" is written $|S|$.

- $|A|$ = ____. $|B|$ = ____. $|C|$ = ____. $|\varnothing|$ = ____.

Sets are unordered, and something can only be an element of a set once. Repeating it when writing out the set has no effect.

- {5, George W. Bush, the planet Mars, my laptop} = C
- {5, George W. Bush, the planet Mars, George W. Bush, 5, my laptop, 5, 5, 5, 5} = C

Elements of sets are objects, not the names of objects (though linguistic strings such as names can be elements of sets). Consequently, it doesn't matter how you name the elements of a set.

- {John Lennon, George Harrison, Paul McCartney, Ringo Starr}
  $\neq$ {"John Lennon", "George Harrison", "Paul McCartney", "Ringo Starr"}

  (the first is a set of four people; the second is a set of four names)

- {the man holding the office of President of the United States in May 2007, this Dell Inspiron, the positive square root of 25, the fourth most distant planet from Sol} = C

Sets can be elements of other sets.

- *D* = {set *A*, set *B*, set *C*}
- *E* = {1, 2, 3, 4, {1, 2}, {3, 4}, {1, 2, 3, 4}} **Question: |*E*| = _____?**

Note: A set X that's a member of another set Y gets counted as a single element when determining the cardinality of Y, regardless of its cardinality. For instance, |*D*| = 3, even though the cardinality of each of its *elements* is greater than 3.

A few more examples:

- *S* = {1, 2, 3, 4, 5, 6, 7} |*S*| = ____
- *S* = {1, 2, 3, 4, 5, {6, 7}} |*S*| = ____
- *S* = {1, 2, {1, 2}, {2}, {3}} |*S*| = ____

This last example may be particularly illuminating: note that {1, 2} is a single element distinct from either 1 or 2; and that {2} is also an element distinct from 2.

2.2. How to specify a set

2.2.1. List Notation

The notation used above. Simply write out a list of the elements of the set.

Advantages
- It's immediately clear what's in the set.
- It works for pretty much any set.

Disadvantages
- The larger the cardinality of the set, the harder it is to write out the full list.
- Once the set is infinite, literally listing the elements becomes impossible. It can still be done using ellipses, but it may be ambiguous:

  F = {1, 3, 5, 7, ...}

  Is *F* the set of odd natural numbers? Or...

2.2.2. Predicate Notation

Describe the members of the set, rather than naming each one. Frequently used notation: "the set of all *x* for which it's true that [condition]" is written {*x* | [condition]}. Examples:
- *A* = the set of Ivy League schools, *or*
  {the Ivy League schools}, *or*
  {*x* | *x* is an Ivy League school}
- *B* = {*x* | *x* is a positive integer}

Advantages

- Concise and unambiguous, both of which are quite handy for infinite sets

Disadvantages

- Doesn't work for every set (good luck expressing *C* this way}
- Tears holes in the space-time continuum...

Russell's Paradox

Let *S* be the set of sets which do not contain themselves, i.e. $S = \{x \mid x \notin x\}$.
Question: is *S* a member of *S*?

(i) If *S* is a not a member of *S*, then it matches the criterion for inclusion in *S*, and therefore it must be a member of *S*.
(ii) If *S* is a member of *S*, then that's because it matches the criterion for inclusion in *S*, i.e. because it's not a member of itself, i.e., *S* is not a member of *S*.

Therefore, *S* is neither a member of *S*, nor not a member of *S*.

Moral of the story: just because you can write it down doesn't mean you *should*. Or, a little more formally: just because you can write it in predicate notation doesn't make it a well-defined set.

2.2.3. Recursive Notation

List some (finite) members of the set, and give a rule for *generating* other elements of the set based on those members. For instance:

(a) $1 \in F$, and
(b) if $x \in F$, then $x + 2 \in F$, and
(c) nothing else is a member of *F*

Important note: this is a *generator*, not a simple description of the contents of the set. (Otherwise, the statement "nothing else is a member of..." wouldn't really help.) For example, consider the two sets:

- $F' = \{1, 3, 5, 7, 9, 11, 13, ...\}$
- $F'' = \{-1, 1, 3, 5, 7, 9, 11, 13, ...\}$

Look at the description of *F* in the box above. It's true that $1 \in F'$ and $1 \in F''$. It's also true that, for all $x \in F'$, $x + 2 \in F'$, and for all $x \in F''$, $x + 2 \in F''$. So which one of these sets is *F*?

Answer: $F = F'$. Recursive notation doesn't simply describe a set, it specifically generates elements of the set. That is, the rules in the box tell us: 1 is a member of *F*, and thus 1+2 is a member of *F*, and (1+2)+2 is a member of *F*, etc. The intuition is that nothing lets us "jump backward" to include -1.

Another example:

(a) Vermont $\in G$, and
(b) if $x \in G$ and y is a US state and the last letter of x's name is the first letter of y's name, then $y \in G$, and
(c) nothing else is a member of G.

- Start with G = {Vermont}.

- Then add every US state such that the last letter of "Vermont" is the first letter of that state's name—i.e. Texas and Tennessee. Now G = {Vermont, Texas, Tennessee}.

- Again, add every US state whose first letter is the last letter of "Vermont", "Texas", or "Tennessee"—since we already have the T's and there are no E's, that'll just be South Dakota and South Carolina. So G = {Vermont, Texas, Tennessee, South Carolina, South Dakota}.

- Apply the rule once again: G = {Vermont, Texas, Tennessee, South Carolina, South Dakota, Alaska, Alabama, Arizona, Arkansas}.

- One more time...now we're adding every state starting with T, S, E, or A. But it turns out we've already got all those states, so further application of the rule won't change anything.

So we're done, and G = {VT, TX, TN, SC, SD, AK, AL, AZ, AR}. (Remember that it doesn't matter what name we use for the elements—it's the states themselves that are members of G.)

One more example:

(a) $3, 4 \in H$, and
(b) if $x \in H$ and $x + 5 < 30$, then $x + 5 \in H$, and
(c) nothing else is a member of H.

- Start: $H = \{3, 4\}$.

- $3 \in H$ and $3 + 5 < 30$, so $3 + 5 = 8 \in H$. And $4 \in H$ and $4 + 5 < 30$, so $4 + 5 = 9 \in H$. Now: $H = \{3, 4, 8, 9\}$.

- Repeat with 8 and 9: $H = \{3, 4, 8, 9, 13, 14\}$. Again...[repeat a few more times]...and now $H = \{3, 4, 8, 9, 13, 14, 18, 19, 23, 24, 28, 29\}$.

- Repeat with 28: $28 \in H$...but $28 + 5 > 30$. So we're done!

And thus: $H = \{3, 4, 8, 9, 13, 14, 18, 19, 23, 24, 28, 29\}$.

Advantages

- Guaranteed to be well-defined

Disadvantages

- Can be somewhat less perspicuous than other methods.
- Works for even fewer sets than predicate notation.

2.3. *Things to do with sets*

What can we do with sets? We can talk about their elements and cardinality, of course. We can also talk about relations between sets:

Definition: A is **IDENTICAL TO** B, written A = B, iff they have exactly the same members.

Definition: A is a **SUBSET** of B ($A \subseteq B$) iff every element of A is an element of B.
A is a **PROPER SUBSET** of B ($A \subset B$) iff $A \subseteq B$ and $A \neq B$.[1]

- {George W. Bush, the planet Mars} $\subseteq$ *C*
- $\{x \mid x$ is evenly divisible by 6$\} \subseteq \{x \mid x$ is evenly divisible by 2$\}$
- $\{x \mid x$ is evenly divisible by 6$\} \subset \{x \mid x$ is evenly divisible by 2$\}$
- $\{x \mid x$ is evenly divisible by 6$\} \not\subseteq \{x \mid x$ is evenly divisible by 4$\}$

And we have operations on sets:

Definition: The **INTERSECTION** of A and B ($A \cap B$) is the set that contains all and only those elements that are in *both* A *and* B.

- {1, 2, 3, 4, 5, 6} $\cap$ {2, 4, 6, 8, 10, 12} = {2, 4, 6}
- $\{x \mid x$ is an Ivy League school$\} \cap \{x \mid x$ is in the state of New York$\}$ = ________
- $\{x \mid x$ is a square number$\} \cap \{x \mid x$ is a prime number$\}$ = ________

Definition: The **UNION** of A and B ($A \cup B$) is the set that contains all and only those elements that are in A *or* B (or both).

- {1, 2, 3, 4, 5, 6} $\cup$ {2, 4, 6, 8, 10, 12} = {1, 2, 3, 4, 5, 6, 8, 10, 12}
- $\{x \mid x$ is an Ivy League school$\} \cup \{x \mid x$ is in the state of New York$\}$ = ________

Additional notation: $A \cap B \cap C$ is often written as $\bigcap\{A, B, C\}$, and $A \cup B \cup C$ as $\bigcup\{A, B, C\}$.

[1] Also: A is a (**PROPER**) **SUPERSET** of B iff B is a (proper) subset of A, written $B \subseteq A$, $B \subset A$.

Definition: The **DIFFERENCE** of A and B (A – B) is the set of all individuals that are in A and not in B.

- {1, 2, 3, 4, 5, 6} – {2, 4, 6, 8, 10, 12} = {1, 3, 5}

Definition: The **COMPLEMENT** of A (A′) is the set of all individuals that are not in A, with respect to some "universe of discourse" U: that is, U – A.

- With respect to the set of integers: $\{x \mid x \text{ is an even integer}\}' = \{x \mid x \text{ is an odd integer}\}$

Definition: The **POWER SET** of A, $\wp(A)$, is the set of all subsets of A.

- If $H = \{a, b, c\}$, then $\wp(H) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
- $|\wp(S)| = 2^{|S|}$

3. TUPLES

You draw two cards from a 52-card deck.
What is the probability that they form a pair?[2]

Definition: A **SEQUENCE** is a list of objects in a particular order.

- <*a*, *b*, *c*>; <George W. Bush, my laptop, 5, the planet Mars>; <1, 2, 3, 4, ...>
- $\{a, b, c\} = \{c, b, a\}$, but $\langle a, b, c\rangle \neq \langle c, b, a\rangle$
- $\{a, b\} = \{a, b, b\}$, but $\langle a, b\rangle \neq \langle a, b, b\rangle$

Finite sequences are also called **TUPLES**; a sequence with *n* elements is an *n*-tuple. (A 2-tuple is usually called an "ordered pair".)

Definition: The **CARTESIAN PRODUCT** of two sets A and B (A × B) is the set of ordered pairs: $\{\langle x, y\rangle \mid x \in A \text{ and } y \in B\}$

- If $H = \{a, b, c\}$ and $J = \{0, 1\}$, then
  - $H \times J = \{\langle a, 0\rangle, \langle b, 0\rangle, \langle c, 0\rangle, \langle a, 1\rangle, \langle b, 1\rangle, \langle c, 1\rangle\}$
  - $J \times J$ = ____

[2] Answer: 100%. Any two objects form a pair. (This is the kind of joke mathematicians find funny. You've been warned.)

4. RELATIONS

Definition: A **RELATION** is a set of pairs.
The **DOMAIN** of a relation R is $\{x \mid \text{there is some } y \text{ such that } <x, y> \in \mathrm{R}\}$.
The **RANGE** of a relation R is $\{y \mid \text{there is some } x \text{ such that } <x, y> \in \mathrm{R}\}$.

Specifically: If $\mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$, then R is a *relation from A to B*.
If $\mathrm{R} \subseteq \mathrm{A} \times \mathrm{A}$, then R is a *relation in A*.
If $<x, y> \in \mathrm{R}$, also written $\mathrm{R}(x, y)$, $\mathrm{R}xy$, or $x\mathrm{R}y$, then R *holds* between x and y.

For example:

- Let S = {Bart, Homer, Grandpa, Lisa, Maggie, Marge}.
  Define the "is the father of" relation F in S: $F \subseteq S \times S$.

- F = {<Grandpa, Homer>, <Homer, Bart>, <Homer, Lisa>, <Homer, Maggie>}
- The domain of F = {Grandpa, Homer}
- The range of F = {Homer, Bart, Lisa, Maggie}

Definition: The **COMPLEMENT** of a relation $\mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$, written R′, is the set of pairs in $\mathrm{A} \times \mathrm{B}$ that are not in R, i.e. $\{<x, y> \mid <x, y> \notin \mathrm{R}\}$

- F' = {<Homer, Marge>, <Homer, Grandpa>, <Bart, Lisa>, ...}

Definition: The **INVERSE** of a relation $\mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$, written R^{-1}, is the set of pairs in R with their elements reversed, i.e. $\{<y, x> \mid <x, y> \in \mathrm{R}\}$

- F^{-1} = {<Homer, Grandpa>, <Bart, Homer>, <Lisa, Homer>, <Maggie, Homer>}

Note that if $\mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$, then $\mathrm{R}' \subseteq \mathrm{A} \times \mathrm{B}$ and $\mathrm{R}^{-1} \subseteq \mathrm{B} \times \mathrm{A}$.

5. FUNCTIONS

Definition: F is a **FUNCTION** from A to B, written $F : A \rightarrow B$, if

F is a relation from A to B such that:
(a) each element in the domain of F maps to only one element in the range, and
(b) domain(F) = A

(Note: if domain(F) $\subset$ A, then F is called a *partial function*. In general, "function" by itself is used for complete functions only.)

For example: If $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$, then which of the following relations from A to B are functions?

- $P = \{<a, 1>, <b, 2>, <c, 3>\}$
- $Q = \{<a, 1>, <b, 2>\}$
- $R = \{<a, 2>, <a, 3>, <b, 4>, <c, 1>\}$
- $S = \{<a, 3>, <b, 2>, <c, 2>\}$

Further notation and definitions:

- If $<x, y> \in F$, then $F(x) = y$, read "F maps x to y".
  In $F(x) = y$: x is the **ARGUMENT**, y is the **VALUE**.

- If each element in the range of F is mapped to by only one element in the domain—i.e., the converse of (a) in the definition—then F is **ONE-TO-ONE**. (If not, F is MANY-TO-ONE.)
- If range(F) = B—i.e., the converse of (b)—then F is **ONTO** (or a function "onto B"). (If not, F is INTO or a function "into B".)
- If F is one-to-one and onto, F is called a **ONE-TO-ONE CORRESPONDENCE**. (Note that in this case F^{-1} is also a function.)

A function that's neither one-to-one nor onto:
$A = \{a, b, c\}, B = \{0, 1, 2\}, F : A \rightarrow B = \{<a, 0>, <b, 1>, <c, 0>\}$

A function that's one-to-one but not onto:
$A = \{a, b, c\}, B = \{0, 1, 2, 3\}, F : A \rightarrow B = \{<a, 0>, <b, 1>, <c, 2>\}$

A function that's onto but not one-to-one:
$A = \{a, b, c\}, B = \{0, 1\}, F : A \rightarrow B = \{<a, 0>, <b, 1>, <c, 0>\}$

A function that's both one-to-one and onto:
$A = \{a, b, c\}, B = \{0, 1, 2\}, F : A \rightarrow B = \{<a, 0>, <b, 1>, <c, 2>\}$
(note that F looks a lot like the not-onto function above; it matters what set the function is a function into.)