Semantics: Predicate Logic

LING 106
Nov. 28, 2007

Propositional Logic: A Definition

- Σ = {**p**, **q**, **r**, ..., ∧, ∨, →, ~, **[**, **]**}
- *F*, the set of formulas of Propositional Logic, is defined recursively as follows.
  - The propositions **p**, **q**, **r**, ... ∈ *F*.
  - If *x* ∈ *F*, then **[**~*x***]** ∈ *F*.
  - If *x*, *y* ∈ *F*, then **[***x* ∧ *y***]** ∈ *F*.
  - If *x*, *y* ∈ *F*, then **[***x* ∨ *y***]** ∈ *F*.
  - If *x*, *y* ∈ *F*, then **[***x* → *y***]** ∈ *F*.
  - Nothing else is in *F*.

Propositional Logic: Truth Tables

[~*x***]** is true if and only if *x* is false.

p	[~p]
T	F
F	T

it is not the case that **p**

p *and* **q**

p	q	[p ∧ q]
T	T	T
T	F	F
F	T	F
F	F	F

[*x* ∧ *y***]** is true iff *x* and *y* are both true.

[*x* ∨ *y***]** is false iff *x* and *y* are both false.

p	q	[p ∨ q]
T	T	T
T	F	T
F	T	T
F	F	F

p *or* **q**

if **p**, **q**

p	q	[p → q]
T	T	T
T	F	F
F	T	T
F	F	T

[*x* → *y***]** is false iff *x* is true and *y* is false.

More Truth Tables

p = *It's raining.*
q = *John is carrying an umbrella.*
r = *It's snowing.*

If it is raining, John is carrying an umbrella. **[p → q]**
John is not carrying an umbrella. **[~q]**
∴ It is not raining. **[~p]**

p	q	[p → q]	[~q]	[~p]
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Whenever both premises are true...
...the conclusion is true.

Either it is raining or it is snowing. **[p ∨ r]**
It is not snowing. **[~r]**
∴ It is raining. **p**

p	r	[p ∨ r]	[~r]	p
T	T	T	F	T
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

Propositional Logic: A Basis for Semantics

- The good news: propositional logic does a good job of modeling connected propositions.
  - If you've taken LING 106, you can draw FSAs.
  - The exam will be hard, or the exam will not be hard.
  - etc.
- The bad news: it only goes so far.
  - It doesn't quite capture *if*, *and*, etc.
  - We need to know where propositions come from.

The Bad News, Part One

- If *p*, *q* = **[p → q]**. True unless *p* is true and *q* is false.
  - If you fail the exam, you'll fail the class.
    - True in cases where: you fail the exam and you fail the class; you pass the exam and you pass the class; you pass the exam and you fail the class.
  - If Abe Lincoln was a Russian spy, then I'm a professor of linguistics.
    - In fact, I *am* a professor of linguistics. Is this still true?
  - If you're thirsty, there's beer in the fridge.
    - If you're not thirsty and there's no beer in the fridge, is this true?
- Answer: the last one (maybe also the second) aren't the thing being represented by → (aka *material implication*).
  - And that's OK...but it means there are sentences that it looked like we could model, that we in fact can't.

The Bad News, Part One

- *p* and *q* = **[p ∧ q]**. True if *p* is true and *q* is true.
- *p* or *q* = **[p ∨ q]**. True if *p* is true or *q* is true.
  - I started lecturing, turned on my laptop, and came into the room.
    - Really?
  - I wrote a letter to my grandmother yesterday, and six men can fit in the back seat of a Ford.
    - What?
  - Either it's November, or George W. Bush is an Irish Penn student.
    - Well, true, but...?

The Bad News, Part Two

All humans are mortal.
Socrates is a human.
∴ Socrates is mortal.

- That didn't help.
- What we need...
  - **human**: </**hjumən**/, *noun*, 'sapient bipedal mammal'>
  - **mortal**: </**mortəl**/, *adjective*, 'subject to death'>
  - **all humans are mortal**:
    - *Sentence*, made from *noun-phrase* (*article* + ***noun***) and *verb-phrase* (*verb* + ***adjective***).
    - *Proposition*, made from the meaning of **all** (whatever that is), the meaning 'sapient bipedal human', and the meaning 'subject to death'.

And Thus: Predicate Logic

- The intuition:
  - Rather than being about *propositions* (and their connectors)...

    If it is raining, John is carrying an umbrella. **[p → q]**
    John is **not** carrying an umbrella. **[~q]**
    ∴ It is **not** raining. **[~p]**

  - ...the classical, Aristotlean syllogism is about *predicates*.

    All humans **are** mortal.
    Socrates **is a** human.
    ∴ Socrates **is** mortal.
- Note: we'll still need propositional logic; we're just adding things to the system.
- We'll still have the elements of propositional logic in our lexicon: {~, ∨, ∧, →, **[**, **]**} ⊂ Σ.

Predicate Logic: A Definition

- *Socrates is a human* / *The Archon of Ptharn is a squorfle* / *George W. Bush is a Penn student* is "about":
  - Socrates / the Archon of Ptharn / George W. Bush ← Individuals
  - humans / squorfles / Penn students ← Predicates
- *Socrates is a human* means "the predicate *human* holds of the individual *Socrates*".
- {~, ∨, ∧, →, **[**, **]**, **(**, **)**, **A**, **B**, ..., **Z**, **a**, **b**, ..., **z**} ⊂ Σ.
  - Capital letters represent predicates.
  - Lowercase letters represent individuals.
  - *Socrates is a human*: **H(s)**, where

Predicate Logic: A Definition

- *All humans are mortal* / *All squorfles are brachtly* / *All Penn students are from Ireland* is "about":
  - humans / squorfles / Penn students ← Predicates
  - mortalness / brachtliness / Irishness ← Also predicates
- *All humans are mortal* means "the predicate *human* holds of the individual...."
  - But what individual?
- *All humans are mortal* means "the predicate *human* holds of the predicate *mortal*".
  - But what would that even mean?
- We're going to need something else in the system.

Predicate Logic: A Definition

- {~, ∨, ∧, →, **[**, **]**, **(**, **)**, **A**, ..., **Z**, **a**, ..., **w**, *x*, *y*, *z*} ⊂ Σ.
  - **CAPITAL LETTERS** represent predicates.
  - **lowercase letters** represent individuals.
  - *italicized lowercase letters* represent variables over individuals.
  - ...with the note that *x*, *y*, *z* are reserved for variables.
    - If we need more variables, we can always number them: x_1, x_2, x_3, ...
- So *all humans are mortal* can be represented as a connection between **H(*x*)** and **M(*x*)**.
- That'll almost cover it, but we'll need just three more things...

Predicate Logic: A Definition

- $\Sigma = \{\forall, \exists, =, \sim, \vee, \wedge, \rightarrow, [,], (,), \mathbf{A}, ..., \mathbf{Z}, \mathbf{a}, ..., \mathbf{w}, x, y, z\}$
- F, the set of formulas of Predicate Logic, is defined recursively as follows.
  - If a is a predicate and b is an individual*, then $a(b) \in F$.
  - If a, b are individuals*, then $[a = b] \in F$.
  - If $m, n \in F$, then $[\sim m]$, $[m \wedge n]$, $[m \vee n]$, $[m \rightarrow n] \in F$.
  - If $m \in F$ and a is an individual variable, then $[\forall a\ m]$ and $[\exists a\ m] \in F$.
  - Nothing else is in F.

*Note: "individual" = either a constant or a variable.

Predicate Logic: A Definition

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  - If $m \in F$ and a is an individual variable, then $[\forall a\ m]$ and $[\exists a\ m] \in F$.
  - Nothing else is in F.
- Some formulas:
  - $\mathbf{H}(\mathbf{s})$
  - $\mathbf{H}(x)$
  - $[\mathbf{H}(\mathbf{s}) \wedge \mathbf{M}(\mathbf{s})]$
  - $[\mathbf{H}(x) \wedge [\mathbf{s} = x]]$
  - $[\forall x\ \mathbf{H}(\mathbf{s})]$
  - $[\sim[\forall x\ \mathbf{H}(x)]]$
  - $[\exists x\ [\mathbf{H}(x) \wedge [\mathbf{s} =$ ...
- Some non-formulas:
  - $\mathbf{H}$
  - $\mathbf{Hs}$
  - x
  - $[\mathbf{H}(\mathbf{s}) \wedge \mathbf{M}(\mathbf{s}) \wedge \mathbf{P}(\mathbf{s})]$
  - $[\mathbf{H}(\mathbf{s}) = \mathbf{P}(\mathbf{s})]$
  - $[\forall \mathbf{s}\ \mathbf{H}(\mathbf{s})]$

Predicate Logic: One More Definition

- If $m \in F$ and a is a variable, then $[\forall a\ m]$ and $[\exists a\ m] \in F$.
- A variable b appearing in a formula m is **bound** if m is part of one of these two formulas.
  - Examples:
    - In $[\forall x\ \mathbf{H}(x)]$, x is bound.
    - In $[\forall x\ [\mathbf{H}(x) \rightarrow \mathbf{M}(y)]]$, x is bound, and y is not.
    - In $[\exists y\ [\forall x\ [\mathbf{H}(x) \rightarrow \mathbf{M}(y)]]]$, both x and y are bound.
    - In $[[\exists y\ [\forall x\ [\mathbf{H}(x) \rightarrow \mathbf{M}(y)]]] \wedge \mathbf{H}(y)]$, the last y is not bound.
- A formula is a proposition if it contains no unbound variables.

What Does It All Mean?

- $[\sim m]$, $[m \wedge n]$, $[m \vee n]$, $[m \rightarrow n]$ are true if etc.
- If a is a predicate and b is an individual, then $a(b)$ is true if the predicate holds of the individual.
  - e.g., if $\mathbf{H}$ = "is human" and $\mathbf{s}$ = "Socrates", then $\mathbf{H}(\mathbf{s})$ is true.
  - Whether $\mathbf{H}(x)$ is true will depend on who x is.
- $[a = b]$ is true if a and b are the same individual.
- $[\forall a\ m]$ is true if substituting *any* individual for a in m makes m true.
- $[\exists a\ m]$ is true if substituting *some* individual for a

A Very Small World

	Humans (**H**)	Cats (**C**)	Deities (**D**)
Male (**B**)	Socrates (**s**) Plato (**p**)	Mr. Kitty (**k**)	Hades (**h**)
Female (**G**)	Lysistrata (**l**)	Fluffy (**f**)	Athena (**a**) Demeter (**d**)

Mortals (**M**)

- $\mathbf{H}(\mathbf{s})$
- $\mathbf{M}(\mathbf{s})$
- $\mathbf{H}(\mathbf{k})$
- $\mathbf{M}(\mathbf{k})$
- $[\mathbf{H}(\mathbf{s}) \wedge \mathbf{M}(\mathbf{s})]$
- $[\mathbf{H}(\mathbf{k}) \wedge \mathbf{M}(\mathbf{k})]$

- $[\forall x\ \mathbf{H}(x)]$
- $[\exists x\ \mathbf{H}(x)]$
- $[\exists x\ [\mathbf{H}(x) \wedge \mathbf{G}(x)]]$
- $[\exists x\ [\mathbf{H}(x) \wedge \mathbf{C}(x)]]$
- $[\exists x\ [\mathbf{H}(x) \rightarrow \mathbf{C}(x)]]$
- $[\forall x\ [\mathbf{H}(x) \rightarrow \mathbf{M}(x)]]$

- *All humans are mortal*:
  - $[\forall x\ [\mathbf{H}(x) \rightarrow \mathbf{M}(x)]]$
- *Some female is mortal*:
  - $[\exists x\ [\mathbf{G}(x) \wedge \mathbf{M}(x)]]$

The Form of Arguments

All humans are mortal. $[\forall x\ [\mathbf{H}(x) \rightarrow \mathbf{M}(x)]]$
Socrates is a human. $\mathbf{H}(\mathbf{s})$
∴ Socrates is mortal. $\therefore \mathbf{M}(\mathbf{s})$

- We can use something like a truth table in this case...
  - Suppose our possible values for x are **s**, **p**. Then:
- ...or we might want a better system.

H	M	$[\forall x\ [\mathbf{H}x \rightarrow \mathbf{M}x]]$	Hs	Ms
-	-	T	F	F
-	s	T	F	T
-	p	T	F	F
-	sp	T	F	T
s	-	F	T	F
s	s	T	T	T
s	p	F	T	F
s	sp	T	T	T
p	-	F	F	F
p	s	F	F	T
p	p	T	F	F
p	sp	T	F	T
sp	-	F	T	F
sp	s	F	T	T
sp	p	F	T	F
sp	sp	T	T	T