Semantics: Propositional Logic

LING 106
Nov. 26, 2007

Classical Logic

- Consider the following...

  All humans are mortal. ← Premise
  Socrates is a human. ← Premise
  ∴ Socrates is mortal. ← Conclusion

  Symbol note: ∴ is read as "therefore".

- ...and then the following.

  The 36 Penn students in this class are smart.
  ∴ In general, Penn students are smart.

- Which argument is better?
  - Well, perhaps neither inherently, but...

Classical Logic

- Second things first:

  The 36 Penn students in this class are smart.
  ∴ In general, Penn students are smart.

  We've seen thousands of swans; they've all been white.
  ✗ All swans are white.

  The 36 Penn students in this class can draw a FSA.
  ✗ In general, Penn students can draw a FSA.

  Of the 36 Penn students in this class, 6 are freshmen.
  ✗ In general, 1/6 of all Penn students are freshmen.

- Conclusion: whether this kind of argument is valid depends on the particulars.

Cygnus atratus

Classical Logic

- Back to the first:

  All humans are mortal.
  Socrates is a human.
  ∴ Socrates is mortal.

  All squorfles are brachtish.
  The Archon of Ptharn is a squorfle.
  ∴ The Archon of Ptharn is brachtish.

  All Penn students are from Ireland. Premises: false
  George W. Bush is a Penn student. Conclusion: false...and yet?
  ∴ George W. Bush is from Ireland.

- Conclusion: this kind of argument is valid by virtue of its form.

Classical Logic

- Though we have to be careful...

  I'm looking at a picture of Mark Twain.
  Mark Twain is the author of *Tom Sawyer*.
  ∴ I'm looking at a picture of the author of *Tom Sawyer*.

  I'm looking at a picture of someone.
  Someone is the author of *Tom Sawyer*.
  ✗ I'm looking at a picture of the author of *Tom Sawyer*.

  Jimmy Olsen thinks Superman can fly.
  Clark Kent is Superman.
  ✗ Jimmy Olsen thinks Clark Kent can fly.

But Anyway...

- Classical logic: an attempt to classify arguments as valid or invalid based solely on their forms.
- Syntax:
  - Basic elements aren't *particular* words, but *syntactic categories*
    - e.g., **john threw the ball** and **mary likes a cat** are syntactically identical
  - Categories combine to create structures
    - *art. + n = noun-phrase* — $[_{NP}$ **the**$_{art}$ **ball**$_{n}]$
      vb + NP = verb-phrase — $[_{VP}$ **threw**$_{v}$ $[_{NP}$ **the**$_{art}$ **ball**$_{n}]]$
      NP + VP = sentence — $[_{S}$ **john**$_{np}$ $[_{VP}$ **threw**$_{v}$ $[_{NP}$ **the**$_{art}$ **ball**$_{n}]]]$
- Logic:
  - Basic elements aren't *particular* sentences, but *sentence types*
  - Sentence types combine to create arguments
- Logic = a foundation for semantics

Propositional Logic

- Let's start with *propositions*.
- A proposition is a unit expressing a fact...basically, a sentence (in the indicative mood).
  - Some propositions:
    - Snow is white.
    - Socrates is mortal.
    - John will give me the ball.
  - Some non-propositions:
    - Give me the ball.
    - Can I have the ball?

Propositional Logic

- Basic fact about propositions: a proposition may be either **true** or **false**.
- Some true propositions:
  - Mark Twain is the author of *Tom Sawyer*.
  - Philadelphia is in Pennsylvania.
- Some false propositions:
  - George W. Bush is from Ireland.
  - Natural language can be modeled with FSAs.
- Some more propositions:
  - My (i.e., Lance's) middle name is "David".
  - The population of the United States is greater than 303,460,194.

Propositional Logic: A Definition

- Let's get away from particular sentences and into abstract reasoning.
- In fact: let's say that
  - Σ = {**p**, **q**, **r**, ..., ∧, ∨, →, ~, **[**, **]**}
  - **p**, **q**, **r**, ..., are propositions.
  - **[** and **]** are brackets.
  - The remaining four symbols remain undefined for now.
  - Spaces are free.
- Some strings over Σ:
  - **p**
  - **∧**
  - **pqp ~ ∨p[[[q**
  - **[p ∨ q] → ~[q ∧ r]**
- Some strings are better than others...

Propositional Logic: A Definition

- *F*, the set of formulas of Propositional Logic, is defined recursively as follows.
  - The propositions **p**, **q**, **r**, ... ∈ *F*.
  - If *x* ∈ *F*, then **[**~*x***]** ∈ *F*.
  - If *x*, *y* ∈ *F*, then **[***x* ∧ *y***]** ∈ *F*.
  - If *x*, *y* ∈ *F*, then **[***x* ∨ *y***]** ∈ *F*.
  - If *x*, *y* ∈ *F*, then **[***x* → *y***]** ∈ *F*.
  - Nothing else is in *F*.
- **Question**: are the following in *F*?
  - **p** yes
  - **[[p ∨ p] ∨ [q ∨ q]]** yes
  - **[p → ~[q ∨ r]]** no
  - **[p ∨ [p ∨ [p ∨ p]]** no
  - **[~[p → r]]** yes

Propositional Logic: A Definition

- We now have a lexicon and a syntax for propositional logic. Its semantics...
  - Each proposition **p**, **q**, **r**, ... may be true or false.
  - **[**~*x***]** is true if and only if *x* is false.
  - **[***x* ∧ *y***]** is true iff *x* and *y* are both true.
  - **[***x* ∨ *y***]** is false iff *x* and *y* are both false.
  - **[***x* → *y***]** is false iff *x* is true and *y* is false.
- We can represent these facts much more clearly with *truth tables*.

Propositional Logic: Truth Tables

[~*x*] is true if and only if *x* is false.

p	**[~p]**
T	F
F	T

it is not the case that **p**

p *and* **q**

p	**q**	**[p ∧ q]**
T	T	T
T	F	F
F	T	F
F	F	F

[*x* ∧ *y*] is true iff *x* and *y* are both true.

[*x* ∨ *y*] is false iff *x* and *y* are both false.

p	**q**	**[p ∨ q]**
T	T	T
T	F	T
F	T	T
F	F	F

p *or* **q**

if **p**, **q**

p	**q**	**[p → q]**
T	T	T
T	F	F
F	T	T
F	F	T

[*x* → *y*] is false iff *x* is true and *y* is false.

Propositional Logic: "if...then"

if **p**, **q**

p	**q**	**[p → q]**
T	T	T
T	F	F
F	T	T
F	F	T

[*x* → *y*] is false iff *x* is true and *y* is false.

S = "If it is raining, then John is carrying an umbrella."

- The first two lines are pretty clear:
  - In cases where it's raining and John is carrying an umbrella, *S* is true.
  - In cases where it's raining and John is not carrying an umbrella, *S* is false.
- What about the last two?
  - In cases where it's not raining and John is carrying an umbrella, *S* is true. ...why?
  - Because if I say *S*, I'm saying that ***if it's raining***, so-and-so is true. If it's not even raining, am I lying?
  - "If your first name starts with a Q, you're passing this class."
  - "If Abe Lincoln was a Russian spy, then I'm a monkey's uncle."

So What Does This Mean?

- The upshot: we can use these to examine the meaning of certain *forms* of sentence...
  - If it's raining, then it's raining.
  - Either it is raining, or it is not raining.
- ...and certain *forms* of argument.
  - If it is raining, John is carrying an umbrella. John is not carrying an umbrella. Therefore, it is not raining.
  - Either it's raining or it's snowing. But it's not snowing. Therefore, it's raining.

More Truth Tables

p = *It's raining.*

- If it's raining, then it's raining.
  - **[p → p]**

p	**[p → p]**
T	T
F	T

p	**q**	**[p → q]**
T	T	T
T	F	F
F	T	T
F	F	T

- Either it's raining, or it's not raining.
  - **[p ∨ [~p]]**

p	**[~p]**	**[p ∨ [~p]]**
T	F	T
F	T	T

p	**q**	**[p ∨ q]**
T	T	T
T	F	T
F	T	T
F	F	F

- Definition: a **tautology** is a sentence which, solely based on its form, is necessarily true.

More Truth Tables

p = *It's raining.*
q = *John is carrying an umbrella.*
r = *It's snowing.*

If it is raining, John is carrying an umbrella. **[p → q]**
John is not carrying an umbrella. **[~q]**
∴ It is not raining. **[~p]**

p	**q**	**[p → q]**	**[~q]**	**[~p]**
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Whenever both premises are true...
...the conclusion is true.

Either it is raining or it is snowing. **[p ∨ r]**
It is not snowing. **[~r]**
∴ It is raining. **p**

p	**r**	**[p ∨ r]**	**[~r]**	**p**
T	T	T	F	T
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

Aside: *Really* Formal Logic

- We've taken a shortcut around a whole lot of the formal bits.
- For example, to prove that **[p → p]** is a tautology, Alonzo Church (p. 81) defines three axioms and two rules of inference, and then offers:

[[s → [p → q]] → [[s → p] → [s → q]]]	1. Axiom #2
[[s → [r → q]] → [[s → r] → [s → q]]]	2. Rule of substitution (**r** for **p**)
[[s → [r → p]] → [[s → r] → [s → p]]]	3. Rule of substitution (**p** for **q**)
[[p → [r → p]] → [[p → r] → [p → p]]]	4. Rule of substitution (**p** for **s**)
[[p → [q → p]] → [[p → q] → [p → p]]]	5. Rule of substitution (**q** for **r**)
[p → [q → p]]	6. Axiom #1
[[p → q] → [p → p]]	7. Rule of *modus ponens* (6 + 5)
[[p → [q → p]] → [p → p]]	8. Rule of substitution (**[q → p]** for **q**)
[p → p]	9. Rule of *modus ponens* (6 + 8)

- We're not going to do that here.

Propositional Logic: A Basis for Semantics

- The good news: propositional logic does a good job of modeling connected propositions.
  - If you've taken LING 106, you can draw FSAs.
  - The exam will be hard, or the exam will not be hard.
  - etc.
- The bad news: it only goes so far.
  - It doesn't quite capture *if*, *and*, etc.
  - We need to know where propositions come from.

The Bad News, Part One

- If *p*, *q* = **[p → q]**. True unless *p* is true and *q* is false.
  - If you fail the exam, you'll fail the class.
    - True in cases where: you fail the exam and you fail the class; you pass the exam and you pass the class; you pass the exam and you fail the class.
  - If Abe Lincoln was a Russian spy, then I'm a professor of linguistics.
    - In fact, I *am* a photographer's brother-in-law. Is this still true?
  - If you're thirsty, there's beer in the fridge.
    - If you're not thirsty and there's no beer in the fridge, is this true?

The Bad News, Part One

- *p* and *q* = **[p ∧ q]**. True if *p* is true and *q* is true.
- *p* or *q* = **[p ∨ q]**. True if *p* is true or *q* is true.
  - I started lecturing, turned on my laptop, and came into the room.
    - Really?
  - I wrote a letter to my grandmother yesterday, and six men can fit in the back seat of a Ford.
    - What?
  - Either it's November, or George W. Bush is an Irish Penn student.
    - Well, true, but...?

The Bad News, Part Two

All humans are mortal.	**p**
Socrates is a human.	**q**
∴ Socrates is mortal.	**r**

- That didn't help.
- What we need...
  - **human**: </**hjumən**/, *noun*, 'sapient bipedal mammal'>
  - **mortal**: </**mortəl**/, *adjective*, 'subject to death'>
  - **all humans are mortal**:
    - *Sentence*, made from *noun-phrase* (*article* + ***noun***) and *verb-phrase* (*verb* + ***adjective***).
    - *Proposition*, made from the meaning of **all** (whatever that is), the meaning 'sapient bipedal human', and the meaning 'subject to death'.
- Next time: we get closer.