Predicate Logic

Besides keeping the connectives from PL, Predicate Logic (PrL) decomposes simple statements into smaller parts: predicates, terms and quantifiers.

(0) John is tall.
   \( T(j) \)

(1) John is taller than Bill.
   \( T(j,b) \)

(2) Everybody sleeps.
   \( \forall x \ [S(x)] \)

(3) Somebody likes David.
   \( \exists x \ [L(x,d)] \)

1. Syntax of PrL.

- Primitive vocabulary:
  (4) Lexical entries, with a denotation of their own:
    a. A set of individual constants, represented with the letters \( a, b, c, d \ldots \)
    b. A set of individual variables \( x_0, x_1, x_2, \ldots y_0, y_1, y_2, \ldots \) Individual constants and individual variables together constitute the set of terms.
    c. A set of predicates, each with a fixed n-arity, represented by \( P, Q, R \ldots \)
  (5) Symbols treated syncategorematically:
    a. The PL logical connectives.
    b. The quantifier symbols \( \exists \) and \( \forall \).

- Syntactic rules:
  (6) a. If \( P \) is a n-ary predicate and \( t_1 \ldots t_n \) are all terms, then \( P(t_1 \ldots t_n) \) is an atomic formula.
    b. If \( \phi \) is a formula, then \( \neg \phi \) is a formula.
    c. If \( \phi \) and \( \psi \) are formulae, then \( (\phi \land \psi) \) are formulae too.
       \( (\phi \lor \psi) \)
       \( (\phi \to \psi) \)
       \( (\phi \leftrightarrow \psi) \)
    d. If \( \phi \) is a formula and \( v \) is a variable, then \( \forall v \phi \) are formulae too.
       \( \exists v \phi \)
    e. Nothing else is a formula in PrL.
  (7) \( \exists x \ L(x,d) \) (7.d.\( \exists \))
      \( \forall x \ L(x,d) \) (7.a)
QUESTION 1: Draw the syntactic tree for the expressions in (8) that are well-formed formulae of PrL.

(8)  
   a. \( \exists \forall (Qa \rightarrow PR(b)(c)) \)
   b. \( \forall x (P(x) \rightarrow \exists y Q(x,y)) \)
   c. \( \exists x_1 \forall x_2 (P(x_1,x_2) \rightarrow (R(x_1) \land Q(x_2,a))) \)

QUESTION 2: Translate into PrL the following English sentences:

(9)  
   a. John likes Susan. [Mostly from GAMUT]
   b. John has a cat that he spoils.
   c. Everything is bitter or sweet.
   d. Either everything is bitter or everything is sweet.
   e. There is something that everybody told Mary.
   f. Everybody told Mary something.
   g. If all logicians are smart, then Alfred is smart too.
   h. Nobody came.
   i. Nobody is loved by no one.
   j. A whale is a mammal.
   k. Barking dogs don’t bite.
   l. Every student that bought a cat took it to the doctor.
   m. Someone who promises something to somebody should do it.

■ Some syntactic notions:

(10) If x is any variable and \( \varphi \) is a formula to which a quantifier has been attached by rule (7.d) to produce \( \forall x \varphi \) or \( \exists x \varphi \), then we say that \( \varphi \) is the scope of the attached quantifier and that \( \varphi \) or any part of \( \varphi \) lies in the scope of that quantifier.

(11) An occurrence of a variable x is bound if it occurs in the scope of \( \forall x \) or \( \exists x \). A variable is free if it is not bound.

(12) Formulae with no free variables are called closed formulae, formulae (simpliciter) or sentences.
Formulae containing a free variable are called open formulae or propositional functions.
2. Semantics of PrL.

- Trial one:
  (10) a. If $\alpha$ is a constant (excluding syncategorematically treated symbols), then $[[\alpha]]$ is specified by a function $F$ (in the Model $M$) that assigns set-theoretical objects to each constant (this is like saying that the semantic value of those constants is fixed in the Lexikon).
  b. If $\alpha$ is a variable, then $????

(14) a. If $P$ is a n-ary predicate and $t_1… t_n$ are all terms, then, for any $s$, $[[P(t_1… t_n)]]^s = 1$ iff $\langle [[[t_1]]^s,…,[[t_n]]^s \rangle \in [[P]]^s$

If $\phi$ and $\psi$ are formulae, then, for any situation $s$,

b. $[[\neg \phi]]^s = 1$ iff $[[\phi]]^s = 0$

c. $[[\phi\land \psi]]^s = 1$ iff $[[\phi]]^s = 1$ and $[[\psi]]^s = 1$

$[[\phi\lor \psi]]^s = 1$ iff $[[\phi]]^s = 1$ or $[[\psi]]^s = 1$

$[[\phi\rightarrow \psi]]^s = 1$ iff $[[\phi]]^s = 0$ or $[[\psi]]^s = 1$

$[[\phi\leftrightarrow \psi]]^s = 1$ iff $[[\phi]]^s = [[\psi]]^s$

d. If $\phi$ is a formula and $v$ is a variable, then, for any situation $s$,

$[[\forall v \phi]]^s = 1$ iff $[[[c/v] \phi]]^s = 1$ for all constants $c$.

$[[\exists v \phi]]^s = 1$ iff $[[[c/v] \phi]]^s = 1$ for some constant $c$.

(15) $[c/v] \phi$ reads as "the formula resulting from having the constant $c$ instead of the variable $v$ in $\phi"$

QUESTION 3: Let us take the situation $s$ depicted in (16). Let us take a language PrL$_1$ such that: the constants $a$, $b$, and $c$ denote the individuals $\blacksquare$, $\bullet$, and $\lozenge$, respectively, the unary predicate $A$ denotes the set of individuals with a circle around, and the binary predicate $R$ denotes the relation encoded by the arrows. [Modified from GAMUT]

(16)

\[
\begin{array}{ccc}
\blacksquare & \bullet \\
\hline
\lozenge
\end{array}
\]

Determine the truth value of the following formulae of PrL$_1$ in $s$, justifying it in detail.

(17) a. $\exists x \exists y \exists z \ ( R(x, y) \land A(y) \land R(x, z) \land \neg A(z) )$

b. $\forall x \ ( R(x, x) )$

c. $\forall x \ ( R(x, x) \leftrightarrow \neg A(x) )$

d. $\exists x \exists y \ ( R(x, y) \land \neg A(x) \land \neg A(y) )$

QUESTION 4: What problem(s) do the semantic rules in (14d) present?
Trial two:

(18) Variable assignments: $g$: set of variables $\rightarrow$ universe of individuals, $D_e$

\[ [[\ ]^{s,g} \]

(13') a. If $\alpha$ is a constant (excluding syncategorematically treated symbols), then $[[\alpha]]^{s,g}$ is specified in the Lexikon for each $s$.

b. If $\alpha$ is a variable, then $[[\alpha]]^{s,g} = g(\alpha)$

(19) $g^{d/v}$ reads as "the variable assignment $g'$ that is exactly like $g$ except (maybe) for $g(v)$, which equals the individual $d$".

(20) QUESTION: Complete the equivalences:

\[
\begin{align*}
g(x) &= Mary & g_{Paul/x}(x) &= g_{Paul/x}(x) = g_{Paul/x}x (x) = g_{Paul/x}x (y) = \\
g(y) &= Susan & g_{Paul/x}(y) &= g_{Paul/x}x (y) = g_{Paul/x}y (y) = \\
\end{align*}
\]

(14') a. If $P$ is a $n$-ary predicate and $t_1 \ldots t_n$ are all terms, then, for any $s$, $[[P(t_1 \ldots t_n)]]^{s,g} = 1$ iff $<[[t_1]]^{s,g}, \ldots,[[t_n]]^{s,g}> \in [[P]]^{s,g}$

If $\phi$ and $\psi$ are formulae, then, for any situation $s$,

b. $[[\neg \phi]]^{s,g} = 1$ iff $[[\phi]]^{s,g} = 0$

c. $[[\phi \land \psi]]^{s,g} = 1$ iff $[[\phi]]^{s,g} = 1$ and $[[\psi]]^{s,g} = 1$

d. $[[\phi \lor \psi]]^{s,g} = 1$ iff $[[\phi]]^{s,g} = 1$ or $[[\psi]]^{s,g} = 1$

e. $[[\phi \rightarrow \psi]]^{s,g} = 1$ iff $[[\phi]]^{s,g} = 0$ or $[[\psi]]^{s,g} = 1$

\[
\begin{align*}
[[\forall \phi]]^{s,g} &= 1 \text{ iff } [[\phi]]^{s,g} = 1 \text{ for all the } d \in D_e. \\
[[\exists \phi]]^{s,g} &= 1 \text{ iff } [[\phi]]^{s,g} = 1 \text{ for some } d \in D_e.
\end{align*}
\]

(14") For any formula $\phi$, $[[\phi]]^s = 1$ iff, for all assignments $g$, $[[\phi]]^{s,g} = 1$.

QUESTION 5 (for home): Take again the situation $s$ described in (16), repeated below. Determine the truth value of the following formulae in $s$, explaining in detail how you followed the new strategy ("trial two") to find it out.

(16)

\[
\begin{align*}
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]

(21) a. $\forall x \ (R(x,x) \rightarrow \exists y \ (R(x,y) \land A(y)))$

b. $\forall x \ (A(x) \rightarrow \exists y \ (R(x,y)))$

c. $\exists x \exists y \ (R(x,y) \land \neg R(y,x) \land \exists z \ (R(x,z) \land R(z,y)))$

d. $\exists x \exists y \exists z \exists u \ (R(z,x) \land R(u,y) \land A(z) \land \neg A(u))$

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Compositional interpretation:

(22) Situation s’, lexical meanings and assignment g:
Three individuals --Allan, Betty and Connor-- named a, b, and c, respectively.
The set {Betty, Connor}, denoted by the unary predicate B.
The relation {<Allan, Allan>, <Allan, Betty>, <Betty, Allan>, <Connor, Betty>},
denoted by the binary predicate S.
Assignment g’: g'(x)=Allan, g'(y)=Betty.

(23) $\exists x (B(x))$

<table>
<thead>
<tr>
<th>$\exists x (B(x))$</th>
<th>(6.d.$\exists$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>$B(x)$</td>
</tr>
<tr>
<td>B</td>
<td>x</td>
</tr>
</tbody>
</table>

$[[B(x)]]^{s’,g} = \{\text{Betty, Connor}\}$

$[[B]]^{s’,g'} = \{\text{Allan}\}$

If we do the same computation with respect to any other assignment g, $[[\exists x B(x)]]^{s'} = 1$. Hence, $[[\exists x B(x)]]^{s'} = 1$ (14').

QUESTION (for home): Draw the syntactic trees and spell out the compositional semantic interpretation of the following PrL formulae:

(24) a. $\forall y (B(y))$
    b. $\exists x (B(a))$
    c. $\forall x (S(y,x))$
    d. $\forall x \exists y (S(x,y))$
    e. $\exists y \forall x (S(x,y))$
3. Some equivalences [From Partee et al.]

For any predicate \( \pi \) and formula \( \phi \):

(25) Law of Quantifier Negation:
\[
\neg \forall x \ (\pi(x)) \iff \exists x \ (\neg \pi(x))
\]
[And, by \( \neg \neg \phi \iff \phi \), also:
\[
\forall x \ (\pi(x)) \iff \neg \exists x \ (\neg \pi(x))
\]
\[
\neg \forall x \ (\neg \pi(x)) \iff \exists x \ (\pi(x))
\]
\[
\forall x \ (\neg \pi(x)) \iff \neg \exists x \ (\pi(x))
\]

(26) Laws of Quantifier (In)Dependence:

a. \( \forall x \forall y \ (\pi(x,y)) \iff \forall y \forall x \ (\pi(x,y)) \)
b. \( \exists x \exists y \ (\pi(x,y)) \iff \exists y \exists x \ (\pi(x,y)) \)
c. \( \exists x \forall y \ (\pi(x,y)) \Rightarrow \forall y \exists x \ (\pi(x,y)) \)

d. \( \forall x \ (\pi(x)) \land \forall x \ (\rho(x)) \iff \exists x \ (\pi(x)) \land \exists x \ (\rho(x)) \)

(27) Laws of Quantifier Distribution:

a. \( \forall x \ (\pi(x) \land \rho(x)) \iff \forall x \ (\pi(x)) \land \forall x \ (\rho(x)) \)
b. \( \exists x \ (\pi(x) \lor \rho(x)) \iff \exists x \ (\pi(x)) \lor \exists x \ (\rho(x)) \)
c. \( \forall x \ (\pi(x)) \lor \forall x \ (\rho(x)) \Rightarrow \forall x \ (\pi(x) \lor \rho(x)) \)
d. \( \exists x \ (\pi(x) \land \rho(x)) \Rightarrow \exists x \ (\pi(x)) \land \exists x \ (\rho(x)) \)

(28) Laws of Quantifier Movement:

a. \( \phi \rightarrow \forall x \ (\pi(x)) \iff \forall x \ (\phi \rightarrow \pi(x)) \)
provided that \( x \) is not free in \( \phi \).
b. \( \phi \rightarrow \exists x \ (\pi(x)) \iff \exists x \ (\phi \rightarrow \pi(x)) \)
provided that \( x \) is not free in \( \phi \) and that someone exists.
c. \( \forall x \ (\pi(x)) \rightarrow \phi \iff \exists x \ (\pi(x) \rightarrow \phi) \)
provided that \( x \) is not free in \( \phi \) and that someone exists.
d. \( \exists x \ (\pi(x)) \rightarrow \phi \iff \forall x \ (\pi(x) \rightarrow \phi) \)
provided that \( x \) is not free in \( \phi \).

(29) Example:
\[
[\exists x (P(x)) \rightarrow \forall y (Q(y))] \iff \forall y \ [\exists x (P(x)) \rightarrow (Q(y))] \quad (27.a)
\]
\[
\iff \forall y \ \forall x \ [(P(x)) \rightarrow (Q(y))] \quad (27.d)
\]

QUESTION: give equivalent or entailed formulae for (29) and (30):

(30) \( \neg [(P(a)) \rightarrow \exists y (Q(y))] \iff \)

(31) \( \exists x \ [(P(x)) \rightarrow \forall y (Q(y))] \iff \)