Pregroup Grammars are Turing Equivalent

In Categorial Grammar, checking to see whether a sentence is grammatical amounts to determining whether the categories of the constituent words multiply out to a designated element d. A natural perspective to take is that the operations on the categories are universal, and that languages differ only in which categories are assigned to lexical items. With $(x/y) \cdot y = x$ as the only equation, the recognized languages over the term–algebra are exactly the context free languages, under the assumption that the categories are drawn from a free algebra. It has been observed that this does not change if we allow associativity and left–inverse, which gives rise to the notion of a pregroup grammar. This paper shows that if we allow arbitrary algebras as valuation algebras these systems have universal (Turing) generative power. In general, let $\langle H, \cdot \rangle$ be a partial algebra and A an alphabet. Put $A_\varepsilon := A \cup \{\varepsilon\}$. We generally assume H has a unit 1 so that $x \cdot 1 = 1 \cdot x = x$. A **valuation** is a relation $h \subseteq A_\varepsilon \times H$ such that each letter (or ε) relates to only finitely many elements of H. We assume that $\langle \varepsilon, 1 \rangle \in h$. If $d \in H$, $L_d(h)$ is the set of strings $\vec{x}$ such that (1) there is a term t in the language over A_ε with $^\frown$ such that the yield of t is $\vec{x}$, and (2) for each instance of a letter or ε in t we can choose a h–related y_i (possibly distinct per distinct occurrences) such that the term evaluates to d if $^\frown$ is now read as $\cdot$. The roundabout definition allows for a string to have any structural analysis, with empty strings inserted at arbitrary points. Let h' be a valuation into $\langle H', \cdot' \rangle$. Put $h \times h' := \{\langle a, y, z \rangle : \langle a, x \rangle \in h, \langle a, y \rangle \in h', a \in A_\varepsilon\}$.

Theorem 1 $L_{\langle d, d' \rangle}(h \times h') = L_d(h) \cap L_{d'}(h')$.

Consequently, any intersection of context free languages is recognizable by some pregroup.

Theorem 2 *Assume H has right or left inverses. Suppose that $L = L_d(h)$, and $\pi : A \to B^*$ a homomorphism. Then there is a valuation h' and d' such that $\pi[L] = L_{d'}(H')$.*

The idea is that if $\pi(a)$ is a letter, it is assigned the same values as a. If $\pi(a)$ is a sequence, say $b_1 b_2$, then we assign to b_1 the values xy^r and y to b_2, where x is a value of a and y is a formal element adjoined to H. Any r.e. language is the homomorphic image of an intersection of two context free languages. We thus get

Theorem 3 *Any r.e. language is the language accepted by some pregroup valuation.*